Nagoya Math. J., to appear

Localized Morrey-Campanato Spaces on Metric Measure Spaces and Applications to Schrödinger Operators

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Abstract. Let $\mathcal{X}$ be a space of homogeneous type in the sense of Coifman and Weiss and $\mathcal{D}$ a collection of balls in $\mathcal{X}$. The authors introduce the localized atomic Hardy space $H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$ with $p\in(0,1]$ and $q\in[1,\infty]\cap(p,\infty]$, the localized Morrey-Campanato space $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ and the localized Morrey-Campanato-BLO space $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ with $\alpha\in\mathbb{R}$ and $p\in(0,\infty)$ and establish their basic properties including $H^{p,\,q}_{\mathcal{D}}(\mathcal{X})=H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ and several equivalent characterizations for $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$. Especially, the authors prove that when $\alpha>0$ and $p\in[1,\infty)$, then $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})=\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})=\mathrm{Lip}_{\mathcal{D}}(\alpha;\,\mathcal{X})$, and when $p\in(0,1]$, then the dual space of $H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ is $\mathcal{E}^{1/p-1,\,1}_{\mathcal{D}}(\mathcal{X})$. Let ρ be an admissible function modeled on the known auxiliary function determined by the Schrödinger operator. Denote the spaces $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$, respectively, by $\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\rho}(\mathcal{X})$, when $\mathcal{D}$ is determined by ρ. The authors then obtain the boundedness from $\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})$ to $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\rho}(\mathcal{X})$ of the radial and the Poisson semigroup maximal functions and the Littlewood-Paley g-function which are defined via kernels modeled on the semigroup generated by the Schrödinger operator. These results apply in a wide range of settings, for instance, to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, or the sub-Laplace Schrödinger operator on Heisenberg groups or connected and simply connected nilpotent Lie groups.

1 Introduction

The theory of Morrey-Campanato spaces plays an important role in harmonic analysis and partial differential equations; see, for example, [1, 27, 28, 29, 25, 19, 17, 18, 5] and their references. It is well-known that the dual space of the Hardy space $H^p(\mathbb{R}^d)$ with $p\in(0,\,1)$ is the Morrey-Campanato space $\mathcal{E}^{1/p-1,\,1}(\mathbb{R}^d)$. Notice that Morrey-Campanato spaces on $\mathbb{R}^d$ are essentially related to the Laplacian Δ, where $\Delta\equiv\sum_{j=1}^d\frac{\partial^2}{\partial x_j^2}$.

On the other hand, there exists an increasing interest on the study of Schrödinger operators on $\mathbb{R}^d$ and the sub-Laplace Schrödinger operators on connected and simply connected nilpotent Lie groups with nonnegative potentials satisfying the reverse Hölder inequality; see, for example, [10, 34, 24, 20, 6, 9, 21, 33, 16]. Let $\mathcal{L}\equiv-\Delta+V$ be the Schrödinger

2000 *Mathematics Subject Classification.* Primary 42B25; Secondary 42B35, 42B30.

Key words and phrases. Space of homogeneous type, Heisenberg group, connected and simply connected nilpotent Lie group, admissible function, Schrödinger operator, atomic Hardy space, Morrey-Campanato space, maximal function, g-function.

The first author is supported by the National Natural Science Foundation (Grant No. 10871025) of China.

operator on $\mathbb{R}^d$, where the potential V is a nonnegative locally integrable function. Denote by $\mathcal{B}_q(\mathbb{R}^d)$ the class of functions satisfying the reverse Hölder inequality of order q. For $V \in \mathcal{B}_{d/2}(\mathbb{R}^d)$ with $d \geq 3$, Dziubański et al [6, 7, 9] studied the BMO-type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$ and the Hardy space $H^p_{\mathcal{L}}(\mathbb{R}^d)$ with $p \in (d/(d+1), 1]$ and, especially, proved that the dual space of $H^1_{\mathcal{L}}(\mathbb{R}^d)$ is $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^d)$. Moreover, they obtained the boundedness on these spaces of the variants of several classical operators, including the radial maximal function and the Littlewood-Paley g-function associated to $\mathcal{L}$. Recently, Huang and Liu [16] further proved that the dual space of $H^p_{\mathcal{L}}(\mathbb{R}^d)$ is certain Morrey-Campanato space. Let $\mathcal{X}$ be an RD-space in [12], which means that $\mathcal{X}$ is a space of homogeneous type in the sense of Coifman and Weiss [3, 4] with the additional property that a reverse doubling condition holds. Let ρ be a given admissible function modeled on the known auxiliary function determined by $V \in \mathcal{B}_{d/2}(\mathbb{R}^d)$ (see [33] or (2.3) below). Then the localized Hardy space $H^1_\rho(\mathcal{X})$, the BMO-type space $\mathrm{BMO}_\rho(\mathcal{X})$ and the BLO-type space $\mathrm{BLO}_\rho(\mathcal{X})$ were introduced and studied by the authors of this paper in [33, 32]. Moreover, the boundedness from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$ of several maximal operators and the Littlewood-Paley g-function, which are defined via kernels modeled on the semigroup generated by the Schrödinger operator, was obtained in [32].

The first purpose of this paper is to investigate behaviors of these operators on localized Morrey-Campanato spaces on metric measure spaces. To be precise, let $\mathcal{X}$ be a space of homogeneous type, which is not necessary to be an RD-space, and $\mathcal{D}$ be a collection of balls in $\mathcal{X}$. In Section 2 of this paper, we first introduce the localized atomic Hardy space $H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$ with $p \in (0,1]$ and $q \in [1,\infty] \cap (p,\infty]$, the localized Morrey-Campanato space $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ and the localized Morrey-Campanato-BLO space $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ with $\alpha \in \mathbb{R}$ and $p \in (0,\infty)$, and establish their basic properties including $H^{p,\,q}_{\mathcal{D}}(\mathcal{X}) = H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ and several equivalent characterizations for $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$. Especially, we prove that when $\alpha > 0$ and $p \in [1,\,\infty)$, then $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X}) = \mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X}) = \mathrm{Lip}_{\mathcal{D}}(\alpha;\,\mathcal{X})$, and when $p \in (0,1]$, then the dual space of $H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ is $\mathcal{E}^{1/p-1,\,1}_{\mathcal{D}}(\mathcal{X})$ (see Theorem 2.1 below). Let ρ be a given admissible function. Modeled on the semigroup generated by the Schrödinger operator, in Sections 3 and 4 of this paper, we introduced the radial maximal operators T^+ and P^+ and Littlewood-Paley g-function $g(\cdot)$. Then we establish the boundedness of T^+ and P^+ from $\mathcal{E}^{\alpha,\,p}_\rho(\mathcal{X})$ to $\widetilde{\mathcal{E}}^{\alpha,\,p}_\rho(\mathcal{X})$ (see Theorems 3.1 and 3.2 below). Here, for the set $\mathcal{D}$ determined by ρ, we denote $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$, respectively, by $\mathcal{E}^{\alpha,\,p}_\rho(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}_\rho(\mathcal{X})$. Moreover, under the assumption that g-function $g(\cdot)$ is bounded on $L^p(\mathcal{X})$ with $p \in (1,\infty)$, we prove that for every $f \in \mathcal{E}^{\alpha,\,p}_\rho(\mathcal{X})$, then $[g(f)]^2 \in \widetilde{\mathcal{E}}^{2\alpha,\,p/2}_\rho(\mathcal{X})$ with norm no more than $C\|f\|^2_{\mathcal{E}^{\alpha,\,p}_\rho(\mathcal{X})}$, where C is a positive constant independent of f (see Theorem 4.1 below). As a simple corollary of this, we obtain the boundedness of $g(\cdot)$ from $\mathcal{E}^{\alpha,\,p}_\rho(\mathcal{X})$ to $\widetilde{\mathcal{E}}^{\alpha,\,p}_\rho(\mathcal{X})$. Notice that $\mathcal{E}^{0,\,p}_\rho(\mathcal{X}) = \mathrm{BMO}_\rho(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{0,\,p}_\rho(\mathcal{X}) = \mathrm{BLO}_\rho(\mathcal{X})$ when $p \in [1,\infty)$. Thus, the results in this section when $\alpha = 0$ and $\mathcal{X}$ is an RD-space were already obtained in [32].

Finally, as the second purpose of this paper, in Section 5 of this paper, we apply results obtained in Sections 3 and 4 of this paper, respectively, to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, the sub-Laplace Schrödinger operator on Heisenberg groups or on connected and simply connected nilpotent Lie groups (see Propositions 5.1 through 5.5 below). The nonnegative potentials of these Schrödinger operators are

assumed to satisfy the reverse Hölder inequality.

We now make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_1 and A_1, do not change in different occurrences. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any given "normed" spaces $\mathcal{A}$ and $\mathcal{B}$, the symbol $\mathcal{A} \subset \mathcal{B}$ means that for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. We also use B to denote a ball of $\mathcal{X}$, and for $\lambda > 0$, λB denotes the ball with the same center as B, but radius λ times the radius of B. Moreover, set $B^{\complement} \equiv \mathcal{X} \setminus B$. Also, for any set $E \subset \mathcal{X}$, χ_E denotes its characteristic function. For all $f \in L^1_{\rm loc}(\mathcal{X})$ and balls B, we always set $f_B \equiv \frac{1}{\mu(B)} \int_B f(y)\, d\mu(y)$.

2 Localized Morrey-Campanato and Hardy spaces

This section is divided into two subsections. In Subsection 2.1, we introduce the localized spaces $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ with $\alpha \in \mathbb{R}$ and $p \in (0,\infty)$, we then establish the relations of these localized spaces with their corresponding global versions and prove that for all $\alpha \in [0,\infty)$ and $p \in (1,\infty)$, $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X}) = \mathcal{E}^{\alpha,\,1}_{\mathcal{D}}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X}) = \widetilde{\mathcal{E}}^{\alpha,\,1}_{\mathcal{D}}(\mathcal{X})$. In Subsection 2.2, we introduce the localized space $H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$ with $p \in (0,1]$ and $q \in [1,\infty] \cap (p,\infty]$, and show that $H^{p,\,q}_{\mathcal{D}}(\mathcal{X}) = H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ and the dual space of $H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ is $\mathcal{E}^{1/p-1,\,1}_{\mathcal{D}}(\mathcal{X})$.

2.1 Localized Morrey-Campanato spaces

We first recall the notion of spaces of homogeneous type in [3, 4].

Definition 2.1 Let $(\mathcal{X},\, d)$ be a metric space endowed with a regular Borel measure μ such that all balls defined by d have finite and positive measure. For any $x \in \mathcal{X}$ and $r \in (0,\infty)$, set the ball $B(x,r) \equiv \{y \in \mathcal{X} :\ d(x,y) < r\}$. The triple $(\mathcal{X},\, d,\, \mu)$ is called a space of homogeneous type if there exists a constant $A_1 \in [1,\infty)$ such that for all $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$\mu(B(x,2r)) \le A_1 \mu(B(x,r)) \ (\textit{doubling property}). \tag{2.1}$$

From (2.1), it is not difficult to see that there exists positive constants A_2 and n such that for all $x \in \mathcal{X}$, $r \in (0,\infty)$ and $\lambda \in [1,\infty)$,

$$\mu(B(x,\lambda r)) \le A_2 \lambda^n \mu(B(x,r)).$$

In what follows, we always set $V_r(x) \equiv \mu(B(x,\, r))$ and $V(x,\, y) \equiv \mu(B(x,\, d(x,\, y)))$ for all $x,\, y \in \mathcal{X}$ and $r \in (0,\, \infty)$.

Definition 2.2 ([33]) A positive function ρ on $\mathcal{X}$ is said to be admissible if there exist positive constants C_0 and k_0 such that for all $x,\, y \in \mathcal{X}$,

$$\frac{1}{\rho(x)} \le C_0 \frac{1}{\rho(y)} \left(1 + \frac{d(x,\, y)}{\rho(y)}\right)^{k_0}. \tag{2.2}$$

Obviously, if ρ is a constant function, then ρ is admissible. Moreover, let $x_0 \in \mathcal{X}$ be fixed. The function $\rho(y) \equiv (1 + d(x_0, y))^s$ for all $y \in \mathcal{X}$ with $s \in (-\infty, 1)$ also satisfies Definition 2.2 with $k_0 = s/(1-s)$ when $s \in [0, 1)$ and $k_0 = -s$ when $s \in (-\infty, 0)$. Another non-trivial class of admissible functions is given by the well-known reverse Hölder class $\mathcal{B}_q(\mathcal{X}, d, \mu)$, which is written as $\mathcal{B}_q(\mathcal{X})$ for simplicity. Recall that a nonnegative potential V is said to be in $\mathcal{B}_q(\mathcal{X})$ with $q \in (1, \infty]$ if there exists a positive constant C such that for all balls B of $\mathcal{X}$,

$$\left(\frac{1}{|B|}\int_B [V(y)]^q\,dy\right)^{1/q} \le \frac{C}{|B|}\int_B V(y)\,dy$$

with the usual modification made when $q = \infty$. It is known that if $V \in \mathcal{B}_q(\mathcal{X})$ for certain $q \in (1, \infty]$, then V is an $A_\infty(\mathcal{X})$ weight in the sense of Muckenhoupt, and also $V \in \mathcal{B}_{q+\epsilon}(\mathcal{X})$ for some $\epsilon \in (0, \infty)$; see, for example, [25] and [26]. Thus $\mathcal{B}_q(\mathcal{X}) = \cup_{q_1 > q}\mathcal{B}_{q_1}(\mathcal{X})$. For all $V \in \mathcal{B}_q(\mathcal{X})$ with certain $q \in (1, \infty]$ and all $x \in \mathcal{X}$, set

$$\rho(x) \equiv [m(x, V)]^{-1} \equiv \sup\left\{r > 0 :\quad \frac{r^2}{\mu(B(x, r))}\int_{B(x, r)} V(y)\,dy \le 1\right\}; \tag{2.3}$$

see, for example, [24] and also [33]. It was also proved in [33] that ρ in (2.3) is an admissible function if $n \ge 1$, $q > \max\{1, n/2\}$ and $V \in \mathcal{B}_q(\mathcal{X})$.

We now recall the notion of Morrey-Campanato spaces and introduce the definitions of Morrey-Campanato-BLO space and their localized versions.

Definition 2.3 Let $\alpha \in \mathbb{R}$ and $p \in (0, \infty)$.

(i) A function $f \in L^p_{\rm loc}(\mathcal{X})$ is said to be in the Morrey-Campanato space $\mathcal{E}^{\alpha, p}(\mathcal{X})$ if

$$\|f\|_{\mathcal{E}^{\alpha, p}(\mathcal{X})} \equiv \sup_{B \subset \mathcal{X}}\left\{\frac{1}{[\mu(B)]^{1+p\alpha}}\int_B |f(y) - f_B|^p d\mu(y)\right\}^{1/p} < \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$ and $f_B = \frac{1}{\mu(B)}\int_B f(z)\,d\mu(z)$.

(ii) A function $f \in L^p_{\rm loc}(\mathcal{X})$ is said to be in the Morrey-Campanato-BLO space $\widetilde{\mathcal{E}}^{\alpha, p}(\mathcal{X})$ if

$$\|f\|_{\widetilde{\mathcal{E}}^{\alpha, p}(\mathcal{X})} \equiv \sup_{B \subset \mathcal{X}}\left\{\frac{1}{[\mu(B)]^{1+p\alpha}}\int_B \left[f(y) - \operatorname*{essinf}_B f\right]^p d\mu(y)\right\}^{1/p} < \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$.

(iii) Let $\alpha \in (0, \infty)$. A function f on $\mathcal{X}$ is said to be in the Lipschitz space $\mathrm{Lip}(\alpha; \mathcal{X})$ if there exists a nonnegative constant C such that for all $x, y \in \mathcal{X}$ and balls B containing x and y,

$$|f(x) - f(y)| \le C[\mu(B)]^\alpha.$$

The minimal nonnegative constant C as above is called the norm of f in $\mathrm{Lip}(\alpha; \mathcal{X})$ and denoted by $\|f\|_{\mathrm{Lip}(\alpha; \mathcal{X})}$.

Remark 2.1 (i) The space $\mathcal{E}^{\alpha,\,p}(\mathcal{X})$ was first introduced by Campanato in [1] when $\mathcal{X}$ is a bounded subset of $\mathbb{R}^d$ and μ is the d-dimensional Lebesgue measure. When $\alpha = 0$, $\mathcal{E}^{0,\,p}(\mathcal{X})$ is just the space $\mathrm{BMO}^p(\mathcal{X})$ (the space of functions of bounded mean oscillation), and $\mathcal{E}^{0,\,p}(\mathcal{X})$ with $p \in [1,\infty)$ coincides with $\mathrm{BMO}^1(\mathcal{X})$; see [4]. For simplicity, we denote $\mathrm{BMO}^1(\mathcal{X})$ by $\mathrm{BMO}(\mathcal{X})$.

(ii) The space $\widetilde{\mathcal{E}}^{0,\,p}(\mathcal{X})$ is just the space $\mathrm{BLO}^p(\mathcal{X})$ (the space of functions of bounded lower oscillation). By (i) of this remark and the fact that $\mathrm{BLO}^1(\mathcal{X}) \subset \mathrm{BMO}(\mathcal{X})$, it is easy to see that $\widetilde{\mathcal{E}}^{0,\,p}(\mathcal{X})$ with $p \in [1,\infty)$ coincides with $\mathrm{BLO}^1(\mathcal{X})$. For simplicity, we denote $\mathrm{BLO}^1(\mathcal{X})$ by $\mathrm{BLO}(\mathcal{X})$. Recall that $\mathrm{BLO}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{\alpha,\,p}(\mathcal{X})$ are not linear spaces. The space $\mathrm{BLO}(\mathbb{R}^d)$ was first introduced by Coifman and Rochberg [2] and $\widetilde{\mathcal{E}}^{\alpha,\,p}(\mathbb{R}^d)$ was introduced in [14].

(iii) When $\alpha \in \mathbb{R}$ and $p \in [1,\infty)$, $\widetilde{\mathcal{E}}^{\alpha,\,p}(\mathcal{X}) \subset \mathcal{E}^{\alpha,\,p}(\mathcal{X})$. Moreover, when $\alpha \in (0,\,\infty)$ and $p \in [1,\infty)$, we have $\widetilde{\mathcal{E}}^{\alpha,\,p}(\mathcal{X}) = \mathcal{E}^{\alpha,\,p}(\mathcal{X}) = \mathrm{Lip}(\alpha;\,\mathcal{X})$ with equivalent norms. In fact, Macías and Segovia [22] proved that when $\alpha \in (0,\infty)$ and $p \in [1,\infty)$, $\mathcal{E}^{\alpha,\,p}(\mathcal{X}) = \mathrm{Lip}(\alpha;\,\mathcal{X})$. On the other hand, for all $f \in \mathcal{E}^{\alpha,\,p}(\mathcal{X})$ and balls B,

$$\int_B [f(y) - \operatorname*{essinf}_B f]^p\,d\mu(y) \le \int_B \operatorname*{esssup}_{x\in B} |f(y) - f(x)|^p\,d\mu(y) \lesssim \|f\|^p_{\mathrm{Lip}(\alpha;\,\mathcal{X})}[\mu(B)]^{1+p\alpha},$$

which implies that $\|f\|_{\widetilde{\mathcal{E}}^{\alpha,\,p}(\mathcal{X})} \lesssim \|f\|_{\mathrm{Lip}(\alpha;\,\mathcal{X})} \sim \|f\|_{\mathcal{E}^{\alpha,\,p}(\mathcal{X})}$. Thus, $\mathcal{E}^{\alpha,\,p}(\mathcal{X}) \subset \widetilde{\mathcal{E}}^{\alpha,\,p}(\mathcal{X})$ and the claim holds.

Definition 2.4 Let $\mathcal{D}$ be a collection of balls in $\mathcal{X}$, $p \in (0,\infty)$ and $\alpha \in \mathbb{R}$. Denote by B any ball of $\mathcal{X}$.

(i) A function $f \in L^p_{\mathrm{loc}}(\mathcal{X})$ is said to be in the localized Morrey-Campanato space $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ if

$$\begin{aligned}\|f\|_{\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})} \equiv \sup_{B\notin\mathcal{D}} &\left\{\frac{1}{[\mu(B)]^{1+p\alpha}}\int_B |f(y) - f_B|^p d\mu(y)\right\}^{1/p}\\ &+ \sup_{B\in\mathcal{D}}\left\{\frac{1}{[\mu(B)]^{1+p\alpha}}\int_B |f(y)|^p d\mu(y)\right\}^{1/p} < \infty,\end{aligned}$$

where $f_B = \frac{1}{\mu(B)}\int_B f(z)\,d\mu(z)$.

(ii) A function $f \in L^p_{\mathrm{loc}}(\mathcal{X})$ is said to be in the localized Morrey-Campanato-BLO space $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ if

$$\begin{aligned}\|f\|_{\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})} \equiv \sup_{B\notin\mathcal{D}} &\left\{\frac{1}{[\mu(B)]^{1+p\alpha}}\int_B \left[f(y) - \operatorname*{essinf}_B f\right]^p d\mu(y)\right\}^{1/p}\\ &+ \sup_{B\in\mathcal{D}}\left\{\frac{1}{[\mu(B)]^{1+p\alpha}}\int_B |f(y)|^p\,d\mu(y)\right\}^{1/p} < \infty.\end{aligned}$$

(iii) Let $\alpha \in (0,\,\infty)$. A function f on $\mathcal{X}$ is said to be in the localized Lipschitz space $\mathrm{Lip}_{\mathcal{D}}(\alpha;\,\mathcal{X})$ if there exists a nonnegative constant C such that for all $x,\,y \in \mathcal{X}$ and balls B containing x and y with $B \notin \mathcal{D}$,

$$|f(x) - f(y)| \le C[\mu(B)]^\alpha,$$

and that for all balls $B \in \mathcal{D}$, $\|f\|_{L^\infty(B)} \le C[\mu(B)]^\alpha$. The minimal nonnegative constant C as above is called the norm of f in $\mathrm{Lip}_{\mathcal{D}}(\alpha;\, \mathcal{X})$ and denoted by $\|f\|_{\mathrm{Lip}_{\mathcal{D}}(\alpha;\, \mathcal{X})}$.

Remark 2.2 (i) When $\alpha = 0$ and $p \in [1, \infty)$, we denote $\mathcal{E}^{0,\,p}_{\mathcal{D}}(\mathcal{X})$ by $\mathrm{BMO}^p_{\mathcal{D}}(\mathcal{X})$ and $\mathrm{BMO}^1_{\mathcal{D}}(\mathcal{X})$ by $\mathrm{BMO}_{\mathcal{D}}(\mathcal{X})$. And we also denote $\widetilde{\mathcal{E}}^{0,\,p}_{\mathcal{D}}(\mathcal{X})$ by $\mathrm{BLO}^p_{\mathcal{D}}(\mathcal{X})$ and $\widetilde{\mathcal{E}}^{0,\,1}_{\mathcal{D}}(\mathcal{X})$ by $\mathrm{BLO}_{\mathcal{D}}(\mathcal{X})$. The localized BLO space was first introduced in [15] in the setting of $\mathbb{R}^d$ endowed with a nondoubling measure.

(ii) If $\mathcal{X}$ is the Euclidean space $\mathbb{R}^d$ and $\mathcal{D} \equiv \{B(x, r) :\ r \ge 1,\, x \in \mathbb{R}^d\}$, then $\mathrm{BMO}_{\mathcal{D}}(\mathcal{X})$ is just the localized BMO space of Goldberg [11], and $\mathrm{Lip}_{\mathcal{D}}(\alpha;\, \mathcal{X})$ with $\alpha \in (0, 1)$ is just the inhomogeneous Lipschitz space (see also [11]).

(iii) For all $\alpha \in \mathbb{R}$ and $p \in (0, \infty)$, $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X}) \subset \mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X}) \subset \mathcal{E}^{\alpha,\,p}(\mathcal{X})$. For $\alpha \in (0, \infty)$, $\mathrm{Lip}_{\mathcal{D}}(\alpha;\, \mathcal{X}) \subset \mathrm{Lip}(\alpha;\, \mathcal{X})$.

(iv) Let ρ be an admissible function and $\mathcal{D}_\rho \equiv \{B(x, r) :\ x \in \mathcal{X},\ r \ge \rho(x)\}$. In this case, we denote $\mathcal{E}^{\alpha,\,p}_{\mathcal{D}_\rho}(\mathcal{X})$, $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}_\rho}(\mathcal{X})$, $\mathrm{Lip}_{\mathcal{D}_\rho}(\alpha;\, \mathcal{X})$, $\mathrm{BMO}_{\mathcal{D}_\rho}(\mathcal{X})$ and $\mathrm{BLO}_{\mathcal{D}_\rho}(\mathcal{X})$, respectively, by $\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})$, $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\rho}(\mathcal{X})$, $\mathrm{Lip}_{\rho}(\alpha;\, \mathcal{X})$, $\mathrm{BMO}_{\rho}(\mathcal{X})$ and $\mathrm{BLO}_{\rho}(\mathcal{X})$. In [32], the spaces $\mathrm{BMO}_{\rho}(\mathcal{X})$ and $\mathrm{BLO}_{\rho}(\mathcal{X})$ when $\mathcal{X}$ is an RD-space were introduced.

The following results follow from Definitions 2.3 and 2.4.

Lemma 2.1 *Let $\mathcal{D}$ be a collection of balls in $\mathcal{X}$, $p \in [1, \infty)$ and $\alpha \in \mathbb{R}$.*

(i) Then $f \in \mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ if and only if $f \in \mathcal{E}^{\alpha,\,p}(\mathcal{X})$ and $\sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} < \infty$; moreover,

$$\|f\|_{\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})} \sim \|f\|_{\mathcal{E}^{\alpha,\,p}(\mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha}.$$

(ii) Then $f \in \widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$ if and only if $f \in \widetilde{\mathcal{E}}^{\alpha,\,p}(\mathcal{X})$ and $\sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} < \infty$; moreover,

$$\|f\|_{\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})} \sim \|f\|_{\widetilde{\mathcal{E}}^{\alpha,\,p}(\mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha}.$$

(iii) Let $\alpha \in (0, \infty)$. Then $f \in \mathrm{Lip}_{\mathcal{D}}(\alpha;\, \mathcal{X})$ if and only if $f \in \mathrm{Lip}(\alpha;\, \mathcal{X})$ and $\sup_{B\in\mathcal{D}}[\mu(B)]^{-\alpha}\ \|f\|_{L^\infty(B)} < \infty$ or $\sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} < \infty$; moreover,

$$\begin{aligned}\|f\|_{\mathrm{Lip}_{\mathcal{D}}(\alpha;\, \mathcal{X})} &\sim \|f\|_{\mathrm{Lip}(\alpha;\, \mathcal{X})} + \sup_{B\in\mathcal{D}} \|f\|_{L^\infty(B)}[\mu(B)]^{-\alpha}\\ &\sim \|f\|_{\mathrm{Lip}(\alpha;\, \mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha}.\end{aligned}$$

Proof. We first prove (i). If $f \in \mathcal{E}^{\alpha,\,p}(\mathcal{X})$ and $\sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} < \infty$, from Definitions 2.3 and 2.4, it follows that

$$\|f\|_{\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})} \le 2\|f\|_{\mathcal{E}^{\alpha,\,p}(\mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha}. \tag{2.4}$$

Conversely, if $f \in \mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})$, then by the Hölder inequality, we have

$$\|f\|_{\mathcal{E}^{\alpha,\,p}(\mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} \le \|f\|_{\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})} + 2\sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} \le 3\|f\|_{\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})},$$

which together with (2.4) gives (i).

The proofs of (ii) and (iii) are similar. We omit the details, which completes the proof of Lemma 2.1.

Lemma 2.2 *Let $\mathcal{D}$ be a collection of balls in $\mathcal{X}$ and $p \in [1, \infty)$.*

(i) Then $\mathrm{BMO}_{\mathcal{D}}(\mathcal{X}) = \mathrm{BMO}^p_{\mathcal{D}}(\mathcal{X})$ and $\mathrm{BLO}_{\mathcal{D}}(\mathcal{X}) = \mathrm{BLO}^p_{\mathcal{D}}(\mathcal{X})$ with equivalent norms.

(ii) When $\alpha \in (0, \infty)$, $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X}) = \mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X}) = \mathrm{Lip}_{\mathcal{D}}(\alpha;\,\mathcal{X})$ with equivalent norms.

Proof. To prove (i), we first assume that $f \in \mathrm{BMO}^p_{\mathcal{D}}(\mathcal{X})$. Then by the Hölder inequality, we have $f \in \mathrm{BMO}_{\mathcal{D}}(\mathcal{X})$ and $\|f\|_{\mathrm{BMO}_{\mathcal{D}}(\mathcal{X})} \le \|f\|_{\mathrm{BMO}^p_{\mathcal{D}}(\mathcal{X})}$. Conversely, if $f \in \mathrm{BMO}_{\mathcal{D}}(\mathcal{X})$, then from Lemma 2.1 (i) with $\alpha = 0$, Remark 2.1 (i) and Remark 2.2 (iii), it follows that

$$\|f\|_{\mathrm{BMO}^p_{\mathcal{D}}(\mathcal{X})} \lesssim \|f\|_{\mathrm{BMO}^p(\mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B| \lesssim \|f\|_{\mathrm{BMO}_{\mathcal{D}}(\mathcal{X})},$$

which implies that $f \in \mathrm{BMO}^p_{\mathcal{D}}(\mathcal{X})$ and $\|f\|_{\mathrm{BMO}^p_{\mathcal{D}}(\mathcal{X})} \lesssim \|f\|_{\mathrm{BMO}_{\mathcal{D}}(\mathcal{X})}$. Thus $\mathrm{BMO}_{\mathcal{D}}(\mathcal{X}) = \mathrm{BMO}^p_{\mathcal{D}}(\mathcal{X})$ with equivalent norms. The proof for $\mathrm{BLO}_{\mathcal{D}}(\mathcal{X}) = \mathrm{BLO}^p_{\mathcal{D}}(\mathcal{X})$ is similar and we omit the details.

To prove (ii), by Lemma 2.1 and Remark 2.1 (iii), we obtain

$$\begin{aligned}
\|f\|_{\mathcal{E}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})} &\sim \|f\|_{\mathcal{E}^{\alpha,\,p}(\mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} \\
&\sim \|f\|_{\widetilde{\mathcal{E}}^{\alpha,\,p}(\mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} \sim \|f\|_{\widetilde{\mathcal{E}}^{\alpha,\,p}_{\mathcal{D}}(\mathcal{X})} \\
&\sim \|f\|_{\mathrm{Lip}(\alpha;\,\mathcal{X})} + \sup_{B\in\mathcal{D}} |f_B|[\mu(B)]^{-\alpha} \sim \|f\|_{\mathrm{Lip}_{\mathcal{D}}(\alpha;\,\mathcal{X})},
\end{aligned}$$

which implies (ii). This finishes the proof of Lemma 2.2.

The space $\mathcal{X}$ is said to have the reverse doubling property if there exist constants $\kappa \in (0, n]$ and $A_3 \in (0, 1]$ such that for all $x \in \mathcal{X}$, $r \in (0, \operatorname{diam}(\mathcal{X})/2]$ and $\lambda \in [1, \operatorname{diam}(\mathcal{X})/(2r)]$,

$$A_3\lambda^\kappa \mu(B(x, r)) \le \mu(B(x, \lambda r)). \tag{2.5}$$

If $(\mathcal{X}, d, \mu)$ satisfies the conditions (2.1) and (2.5), then $(\mathcal{X}, d, \mu)$ is called an RD-space (see [12]).

Lemma 2.3 *Let $\mathcal{X}$ be an RD-space, ρ an admissible function on $\mathcal{X}$ and $\mathcal{D}_\rho$ as in Remark 2.2 (iv). If $\alpha \in (-\infty, 0)$ and $p \in [1, \infty)$, then*

$$\|f\|_{\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})} \sim \sup_{B\subset\mathcal{X}} \left\{ \frac{1}{[\mu(B)]^{1+\alpha p}} \int_B |f(x)|^p \, d\mu(x) \right\}^{1/p}.$$

Proof. An application of the Hölder inequality leads to that

$$\|f\|_{\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})} \lesssim \sup_{B\subset\mathcal{X}} \left\{ \frac{1}{[\mu(B)]^{1+\alpha p}} \int_B |f(x)|^p \, d\mu(x) \right\}^{1/p}.$$

Conversely, if $B \in \mathcal{D}_\rho$, then by Definition 2.4 (i), we have

$$\left\{ \frac{1}{[\mu(B)]^{1+\alpha p}} \int_B |f(x)|^p \, d\mu(x) \right\}^{1/p} \le \|f\|_{\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})}.$$

Now we assume that $B \equiv B(x_0, r) \notin \mathcal{D}_\rho$. Let $J_0 \in \mathbb{N}$ such that $2^{J_0-1}r < \rho(x_0) \le 2^{J_0}r$. From $\alpha \in (-\infty, 0)$, (2.1), (2.5) and the Hölder inequality, it follows that

$$\begin{aligned}
&\left\{\frac{1}{[\mu(B)]^{1+\alpha p}}\int_B |f(x)|^p\,d\mu(x)\right\}^{1/p}\\
&\quad\le \frac{1}{[\mu(B)]^{\alpha}}\left\{\left[\frac{1}{\mu(B)}\int_B |f(x)-f_B|^p\,d\mu(x)\right]^{1/p} + \sum_{j=1}^{J_0}|f_{2^{j-1}B}-f_{2^jB}| + |f_{2^{J_0}B}|\right\}\\
&\quad\lesssim \left(1+\sum_{j=1}^{J_0}2^{j\kappa\alpha}\right)\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})},
\end{aligned}$$

which completes the proof of Lemma 2.3.

Remark 2.3 Let $\mathcal{X}$ be an RD-space and $p \in [1,\, \infty)$.

(i) Then Lemma 2.3 implies that $\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})$ with $\alpha \in (-1/p, 0)$ coincides with the so-called Morrey space (see, for example, [27, 29] for the case $\mathcal{X} = \mathbb{R}^d$).

(ii) Let $\alpha < 0$. For all $f \ge 0$, $f \in \mathcal{E}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})$ if and only if $f \in \widetilde{\mathcal{E}}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})$ and moreover, $\|f\|_{\widetilde{\mathcal{E}}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})} \sim \|f\|_{\mathcal{E}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})}$. In fact, by Remark 2.2 (iii), we only need to show that for all $f \ge 0$, $f \in \mathcal{E}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})$ implies that $f \in \widetilde{\mathcal{E}}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})$ and $\|f\|_{\widetilde{\mathcal{E}}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})} \lesssim \|f\|_{\mathcal{E}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})}$. By Lemma 2.3, the fact that $\alpha < 0$ and $f \ge 0$, we see that for all balls $B \notin \mathcal{D}$,

$$\int_B \left[f(x) - \operatorname*{essinf}_B f\right]^p\,d\mu(x) \le \int_B [f(x)]^p\,d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}\|f\|^p_{\mathcal{E}_\mathcal{D}^{\alpha,\,p}(\mathcal{X})},$$

which implies the claim.

(iii) If $\mathcal{X}$ is not an RD-space, it is not clear if Lemma 2.3 still holds.

We also have the following conclusions which are used in Sections 3 and 4.

Lemma 2.4 *Let $\alpha \in \mathbb{R}$, $p \in [1,\infty)$, ρ be an admissible function on $\mathcal{X}$ and $\mathcal{D}_\rho$ as in Remark 2.2 (iv). Then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})$,*

(i) for all balls $B \equiv B(x_0,\, r) \notin \mathcal{D}_\rho$,

$$\frac{1}{\mu(B)}\int_B |f(z)|\,d\mu(z) \le \begin{cases} C\left(\frac{\rho(x_0)}{r}\right)^{\alpha n}[\mu(B)]^\alpha\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}, & \alpha > 0,\\ C\left(1+\log\frac{\rho(x_0)}{r}\right)[\mu(B)]^\alpha\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}, & \alpha \le 0;\end{cases}$$

(ii) for all $x \in \mathcal{X}$ and $0 < r_1 < r_2$,

$$|f_{B(x,r_1)} - f_{B(x,r_2)}| \le \begin{cases} C\left(\frac{r_2}{r_1}\right)^{\alpha n}[\mu(B(x,\,r_1)]^\alpha\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}, & \alpha > 0,\\ C\left(1+\log\frac{r_2}{r_1}\right)[\mu(B(x,\,r_1)]^\alpha\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}, & \alpha \le 0.\end{cases}$$

Proof. If (ii) holds, then by the Hölder inequality, we see that for all $f \in \mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})$ and $B \notin \mathcal{D}_\rho$,

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |f(x)|\,d\mu(x) &\le \frac{1}{\mu(B)}\int_B |f(x)-f_B|\,d\mu(x) + \left|f_B - f_{B(x_0,\,\rho(x_0))}\right| \\ &\quad + \frac{1}{\mu(B(x_0,\,\rho(x_0)))}\int_{B(x_0,\,\rho(x_0))} |f(x)|\,d\mu(x) \\ &\lesssim \{[\mu(B)]^\alpha + [\mu(B(x_0,\rho(x_0)))]^\alpha\}\,\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})} + \left|f_B - f_{B(x_0,\,\rho(x_0))}\right|.\end{aligned}$$

Then (i) follows from this fact together with (2.1), $r < \rho(x_0)$ (because $B \notin \mathcal{D}_\rho$) and (ii).

To prove (ii), let j_0 be the smallest integer such that $2^{j_0} r_1 \ge r_2$. Another application of (2.1) leads to that

$$\begin{aligned}\left|f_{B(x,\,2^{j_0}r_1)} - f_{B(x,r_2)}\right| &\lesssim \frac{1}{\mu(B(x,\,2^{j_0}r_1))}\int_{B(x,\,2^{j_0}r_1)} \left|f - f_{B(x,\,2^{j_0}r_1)}\right|\,d\mu(z) \\ &\lesssim \left[\mu(B(x,\,2^{j_0}r_1))\right]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}.\end{aligned}$$

Similarly, we see that for all $j \in \mathbb{N} \cup \{0\}$,

$$\left|f_{B(x,\,2^j r_1)} - f_{B(x,\,2^{j+1}r_1)}\right| \lesssim \left[\mu\left(B\left(x,\,2^{j+1}r_1\right)\right)\right]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}.$$

Then we have

$$\begin{aligned}\left|f_{B(x,r_1)} - f_{B(x,r_2)}\right| &\lesssim \sum_{j=0}^{j_0-1}\left|f_{B(x,\,2^j r_1)} - f_{B(x,\,2^{j+1}r_1)}\right| + \left|f_{B(x,\,2^{j_0}r_1)} - f_{B(x,r_2)}\right| \\ &\lesssim \sum_{j=0}^{j_0-1}[\mu(B(x,\,2^{j+1}r_1))]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}.\end{aligned}$$

If $\alpha \in (-\infty, 0]$, from the choice of j_0, we deduce that

$$|f_{B(x,r_1)} - f_{B(x,r_2)}| \lesssim \left(1 + \log\frac{r_2}{r_1}\right)[\mu(B(x,\,r_1))]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})};$$

if $\alpha \in (0, \infty)$, by (2.1), we obtain that

$$|f_{B(x,r_1)} - f_{B(x,r_2)}| \lesssim \left(\frac{r_2}{r_1}\right)^{\alpha n}[\mu(B(x,\,r_1))]^\alpha \|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}.$$

This finishes the proof of Lemma 2.4.

2.2 Localized Hardy spaces

We begin with the notion of atoms.

Definition 2.5 Let $\mathcal{D}$ be a collection of balls in $\mathcal{X}$, $p \in (0,1]$ and $q \in [1,\infty] \cap (p,\infty]$.

(i) A function a supported in a ball $B \subset \mathcal{X}$ is called a $(p,\, q)$-atom if $\int_{\mathcal{X}} a(x)\, d\mu(x) = 0$ and $\|a\|_{L^q(\mathcal{X})} \le [\mu(B)]^{1/q-1/p}$ (see [4]).

(ii) A function b supported in a ball $B \in \mathcal{D}$ is called a $(p,\, q)_{\mathcal{D}}$-atom if $\|b\|_{L^q(\mathcal{X})} \le [\mu(B)]^{1/q-1/p}$.

Remark 2.4 (i) Every $(1,\, q)$-atom or $(1,\, q)_{\mathcal{D}}$-atom a belongs to $L^1(\mathcal{X})$ with $\|a\|_{L^1(\mathcal{X})} \le 1$.

(ii) Let $p \in (0,1)$. If a is a $(p,\, q)$-atom, then $a \in (\,\mathrm{Lip}(1/p-1;\, \mathcal{X}))^* \subset (\,\mathrm{Lip}_{\mathcal{D}}(1/p-1;\, \mathcal{X}))^*$ and $\|a\|_{(\,\mathrm{Lip}_{\mathcal{D}}(1/p-1;\,\mathcal{X}))^*} \le \|a\|_{(\,\mathrm{Lip}(1/p-1;\,\mathcal{X}))^*} \le 1$; if b is a $(p,\, q)_{\mathcal{D}}$-atom, then $b \in (\,\mathrm{Lip}_{\mathcal{D}}(1/p-1;\, \mathcal{X}))^*$ and $\|b\|_{(\,\mathrm{Lip}_{\mathcal{D}}(1/p-1;\,\mathcal{X}))^*} \le 1$.

Definition 2.6 ([4]) Let $p \in (0,1]$ and $q \in [1,\infty] \cap (p,\infty]$. A function $f \in L^1(\mathcal{X})$ or a linear functional $f \in (\,\mathrm{Lip}(1/p-1;\, \mathcal{X}))^*$ when $p \in (0,1)$ is said to be in the Hardy space $H^{1,\,q}(\mathcal{X})$ when $p=1$ or in $H^{p,\,q}(\mathcal{X})$ when $p \in (0,1)$ if there exist $(p,\, q)$-atoms $\{a_j\}_{j=1}^\infty$ and $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$, which converges in $L^1(\mathcal{X})$ when $p=1$ or in $(\,\mathrm{Lip}(1/p-1;\, \mathcal{X}))^*$ when $p \in (0,1)$, and $\sum_{j\in\mathbb{N}} |\lambda_j|^p < \infty$. Moreover, the norm of f in $H^{p,\,q}(\mathcal{X})$ with $p \in (0,1]$ is defined by

$$\|f\|_{H^{p,\,q}(\mathcal{X})} \equiv \inf\left\{\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p}\right\},$$

where the infimum is taken over all the above decompositions of f.

Remark 2.5 Coifman and Weiss [4] proved that $H^{p,\,q}(\mathcal{X})$ and $H^{p,\,\infty}(\mathcal{X})$ coincide with equivalent norms for all $p \in (0,1]$ and $q \in [1,\, \infty) \cap (p,\, \infty)$. Thus, for all $p,\, q$ in this range, we denote $H^{p,\,q}(\mathcal{X})$ simply by $H^p(\mathcal{X})$. We remark that Coifman and Weiss [4] also proved that the dual space of $H^p(\mathcal{X})$ is $\mathrm{BMO}(\mathcal{X})$ when $p=1$ or $\mathrm{Lip}(1/p-1;\, \mathcal{X})$ when $p \in (0,1)$.

Definition 2.7 Let $\mathcal{D}$ be a collection of balls in $\mathcal{X}$, $p \in (0,1]$ and $q \in [1,\infty] \cap (p,\infty]$. A function $f \in L^1(\mathcal{X})$ or a linear functional $f \in (\,\mathrm{Lip}_{\mathcal{D}}(1/p-1;\, \mathcal{X}))^*$ when $p \in (0,1)$ is said to be in $H^{1,\,q}_{\mathcal{D}}(\mathcal{X})$ when $p=1$ or $H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$ when $p \in (0,1)$ if there exist $\{\lambda_j\}_{j\in\mathbb{N}}$, $\{\nu_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$, $(p,\, q)$-atoms $\{a_j\}_{j\in\mathbb{N}}$ and $(p,\, q)_{\mathcal{D}}$-atoms $\{b_k\}_{k\in\mathbb{N}}$ such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j + \sum_{k\in\mathbb{N}} \nu_k b_k,$$

which converges in $L^1(\mathcal{X})$ when $p=1$ or in $(\,\mathrm{Lip}_{\mathcal{D}}(1/p-1;\, \mathcal{X}))^*$ when $p \in (0,1]$, and $\sum_{j\in\mathbb{N}} |\lambda_j|^p + \sum_{k=1}^\infty |\nu_k|^p < \infty$. Moreover, the norm of f in $H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$ is defined by

$$\|f\|_{H^{p,\,q}_{\mathcal{D}}(\mathcal{X})} \equiv \inf\left\{\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p + \sum_{k\in\mathbb{N}} |\nu_k|^p\right)^{1/p}\right\},$$

where the infimum is taken over all the above decompositions of f.

Remark 2.6 Let $p \in (0,1]$ and $q \in [1,\infty] \cap (p,\infty]$. It is easy to see that $H^{p,\,q}(\mathcal{X}) \subset H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$.

Using Remark 2.6, we have the following conclusion.

Lemma 2.5 *Let $\mathcal{D}$ be a collection of balls in $\mathcal{X}$, $p\in(0,1]$ and $q\in[1,\infty)\cap(p,\infty)$. Then $H^{p,\,q}_{\mathcal{D}}(\mathcal{X})=H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ with equivalent norms.*

Proof. Notice that $(p,\,\infty)$-atoms and $(p,\,\infty)_{\mathcal{D}}$-atoms are $(p,\,q)$-atoms and $(p,\,q)_{\mathcal{D}}$-atoms, respectively. Then from Definition 2.7, it follows that $H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})\subset H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$.

Conversely, let $f\in H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$. Then by Definition 2.7, there exist $\{\lambda_j\}_{j\in\mathbb{N}}$, $\{\nu_k\}_{k\in\mathbb{N}}\subset\mathbb{C}$, $(p,\,q)$-atoms $\{a_j\}_{j\in\mathbb{N}}$ and $(p,\,q)_{\mathcal{D}}$-atoms $\{b_k\}_{k\in\mathbb{N}}$ such that

$$f=\sum_{j\in\mathbb{N}}\lambda_j a_j+\sum_{k\in\mathbb{N}}\nu_k b_k,$$

which converges in $L^1(\mathcal{X})$ when $p=1$ or in $(\,\mathrm{Lip}_{\mathcal{D}}(1/p-1;\,\mathcal{X}))^*$ when $p\in(0,1]$, and

$$\sum_{j\in\mathbb{N}}|\lambda_j|^p+\sum_{k\in\mathbb{N}}|\nu_k|^p\lesssim\|f\|^p_{H^{p,\,q}_{\mathcal{D}}(\mathcal{X})}. \tag{2.6}$$

For $k\in\mathbb{N}$, assume that $\operatorname{supp} b_k\subset B_k\in\mathcal{D}$ and let $c_k\equiv[b_k-(b_k)_{B_k}\chi_{B_k}]/2$. Then it follows from Definition 2.5 that there exists a positive constant $\widetilde{C}$ such that $\{\widetilde{C}c_k\}_{k\in\mathbb{N}}$ are $(p,\,q)$-atoms, $\{(b_k)_{B_k}\chi_{B_k}\}_{k\in\mathbb{N}}$ are $(p,\,\infty)_{\mathcal{D}}$-atoms and $b_k=2c_k+(b_k)_{B_k}\chi_{B_k}$. Moreover,

$$f=\sum_{j\in\mathbb{N}}\lambda_j a_j+\sum_{k\in\mathbb{N}}2\nu_k c_k+\sum_{k\in\mathbb{N}}\nu_k(b_k)_{B_k}\chi_{B_k},$$

which converges in $L^1(\mathcal{X})$ when $p=1$ or in $(\,\mathrm{Lip}_{\mathcal{D}}(1/p-1;\,\mathcal{X}))^*$ when $p\in(0,1)$. By Remark 2.4 (ii) and (2.6), we see that $\sum_{j\in\mathbb{N}}\lambda_j a_j+2\sum_{k\in\mathbb{N}}\nu_k c_k$ also converges in $L^1(\mathcal{X})$ when $p=1$ or in $(\,\mathrm{Lip}(1/p-1;\,\mathcal{X}))^*$ when $p\in(0,1)$. Let $g\equiv\sum_{j\in\mathbb{N}}\lambda_j a_j+2\sum_{k\in\mathbb{N}}\nu_k c_k$. Then Definition 2.6 together with Remark 2.5 implies that $g\in H^{p,\,q}(\mathcal{X})=H^{p,\,\infty}(\mathcal{X})$. Form this, Remark 2.6 and (2.6), we deduce that $g\in H^{p,\,\infty}(\mathcal{X})\subset H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ and

$$\|g\|_{H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})}\lesssim\|g\|_{H^{p,\,\infty}(\mathcal{X})}\lesssim\|g\|_{H^{p,\,q}(\mathcal{X})}\lesssim\|f\|_{H^{p,\,q}_{\mathcal{D}}(\mathcal{X})},$$

which further implies that $f\in H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})$ and by (2.6),

$$\begin{aligned}\|f\|_{H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})}&\lesssim\|g\|_{H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})}+\left\|\sum_{k\in\mathbb{N}}\nu_k(b_k)_{B_k}\chi_{B_k}\right\|_{H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X})}\\&\lesssim\|f\|_{H^{p,\,q}_{\mathcal{D}}(\mathcal{X})}+\left\{\sum_{k\in\mathbb{N}}|\nu_k|^p\right\}^{1/p}\lesssim\|f\|_{H^{p,\,q}_{\mathcal{D}}(\mathcal{X})}.\end{aligned}$$

This finishes the proof of Lemma 2.5.

Remark 2.7 (i) Let $\mathcal{D}$ be a collection of balls in $\mathcal{X}$, $p\in(0,1]$ and $q\in[1,\infty]\cap(p,\infty]$. In what follows, based on Lemma 2.5, we denote $H^{p,\,q}_{\mathcal{D}}(\mathcal{X})$ simply by $H^p_{\mathcal{D}}(\mathcal{X})$.

(ii) Let $L^\infty_b(\mathcal{X})$ be the set of all functions of $L^\infty(\mathcal{X})$ with bounded support. Then from Definitions 2.6 and 2.7, it follows that $L^\infty_b(\mathcal{X})\cap H^p_{\mathcal{D}}(\mathcal{X})$ is dense in $H^p_{\mathcal{D}}(\mathcal{X})$ and $L^\infty_b(\mathcal{X})\cap H^p(\mathcal{X})$ is dense in $H^p(\mathcal{X})$.

Theorem 2.1 *Let $\mathcal{D}$ be a collection of balls in $\mathcal{X}$ and $p \in (0,1]$. Then $\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X}) = (H^p_{\mathcal{D}}(\mathcal{X}))^*$.*

Proof. We first prove $\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X}) \subset (H^{p,\,\infty}_{\mathcal{D}}(\mathcal{X}))^*$ for $p \in (0,\,1]$. Let $f \in \mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})$. For all $(p,\,\infty)$-atoms a supported in $B \notin \mathcal{D}$, by Definition 2.5 (i), we have

$$\begin{aligned}\left|\int_{\mathcal{X}} f(x)a(x)\,d\mu(x)\right| &= \left|\int_{\mathcal{X}} [f(x)-f_B]a(x)\,d\mu(x)\right|\\ &\le \frac{1}{[\mu(B)]^{1/p}}\int_B |f(x)-f_B|\,d\mu(x) \le \|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})}.\end{aligned}$$

For all $(p,\,\infty)_{\mathcal{D}}$-atoms b supported in $B \in \mathcal{D}$, we also obtain

$$\left|\int_{\mathcal{X}} f(x)b(x)\,d\mu(x)\right| \le \frac{1}{[\mu(B)]^{1/p}}\int_B |f(x)|\,d\mu(x) \le \|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})}.$$

Let $N \in \mathbb{N}$ and $f_N \equiv \max\{\min\{f, N\}, -N\}$. We claim that $f_N \in \mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})$ and

$$\|f_N\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})} \le \frac{9}{4}\|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})}. \tag{2.7}$$

In fact, if $B \in \mathcal{D}$, then

$$\frac{1}{[\mu(B)]^{1/p}}\int_B |f_N(x)|\,d\mu(x) \le \frac{1}{[\mu(B)]^{1/p}}\int_B |f(x)|\,d\mu(x) \le \|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})}.$$

Let $B \notin \mathcal{D}$. For all $f,\, h \in \mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})$ and $g \equiv \max\{f,\, h\}$, we have that $g = (f+h+|f-h|)/2$ and

$$\begin{aligned}&\frac{1}{[\mu(B)]^{1/p}}\int_B |g(x)-g_B|\,d\mu(x)\\ &\quad\le \frac{1}{2[\mu(B)]^{1/p}}\int_B |f(x)-f_B|\,d\mu(x) + \frac{1}{2[\mu(B)]^{1/p}}\int_B |h(x)-h_B|\,d\mu(x)\\ &\qquad+\frac{1}{[\mu(B)]^{1/p}}\int_B |(f-h)(x)-(f-h)_B|\,d\mu(x)\\ &\quad\le \frac{3}{2}(\|f\|_{\mathcal{E}^{1/p-1,\,1}(\mathcal{X})} + \|h\|_{\mathcal{E}^{1/p-1,\,1}(\mathcal{X})}).\end{aligned}$$

Similarly, for all $B \notin \mathcal{D}$, $f,\, h \in \mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})$ and $\widetilde{g} \equiv \min\{f,\, h\}$, we have

$$\frac{1}{[\mu(B)]^{1/p}}\int_B |\widetilde{g}(x)-\widetilde{g}_B|\,d\mu(x) \le \frac{3}{2}\left(\|f\|_{\mathcal{E}^{1/p-1,\,1}(\mathcal{X})} + \|h\|_{\mathcal{E}^{1/p-1,\,1}(\mathcal{X})}\right).$$

If $h \equiv N$ or $h \equiv -N$, then $\|h\|_{\mathcal{E}^{1/p-1,\,1}(\mathcal{X})} = 0$. By these facts and the definition of f_N, we have that for all $B \notin \mathcal{D}$,

$$\frac{1}{[\mu(B)]^{1/p}}\int_B |f_N(x)-(f_N)_B|\,d\mu(x) \le \frac{9}{4}\|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})},$$

which implies the claim.

For all $g \in L_b^\infty(\mathcal{X}) \cap H_{\mathcal{D}}^{p,\infty}(\mathcal{X})$, since $fg \in L^1(\mathcal{X})$, we define $\ell(g) \equiv \int_{\mathcal{X}} f(x)g(x)\,d\mu(x)$ and $\ell_N(g) \equiv \int_{\mathcal{X}} f_N(x)g(x)\,d\mu(x)$. Moreover, there exist $\{\lambda_j\}$, $\{\nu_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$, (p,∞)-atoms $\{a_j\}_{j\in\mathbb{N}}$ and $(p,\infty)_{\mathcal{D}}$-atoms $\{b_k\}_{k\in\mathbb{N}}$ such that

$$g = \sum_{j\in\mathbb{N}} \lambda_j a_j + \sum_{k\in\mathbb{N}} \nu_k b_k$$

which converges in $L^1(\mathcal{X})$ when $p=1$ or in $(\mathrm{Lip}_{\mathcal{D}}(1/p-1;\,\mathcal{X}))^*$ when $p \in (0,1)$, and

$$\sum_{j\in\mathbb{N}} |\lambda_j|^p + \sum_{k\in\mathbb{N}} |\nu_k|^p \le 2\|g\|^p_{H_{\mathcal{D}}^{p,\infty}(\mathcal{X})}. \tag{2.8}$$

By $f_N \in \mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})$ and $g \in H_{\mathcal{D}}^{p,\infty}(\mathcal{X})$, we have

$$\ell_N(g) = \sum_{j\in\mathbb{N}} \int_{\mathcal{X}} f_N(x)\lambda_j a_j(x)\,d\mu(x) + \sum_{k\in\mathbb{N}} \int_{\mathcal{X}} f_N(x)\nu_k b_k(x)\,d\mu(x),$$

from which together with (2.7), (2.8) and Remark 2.4 (ii), it follows that

$$|\ell_N(g)| \lesssim \|f_N\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})} \left\{ \sum_{j\in\mathbb{N}} |\lambda_j| + \sum_{k\in\mathbb{N}} |\nu_k| \right\} \lesssim \|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})} \|g\|_{H_{\mathcal{D}}^{p,\infty}(\mathcal{X})}.$$

By this and the Lebesgue dominated theorem, we have

$$|\ell(g)| = \lim_{N\to\infty} \left| \int_{\mathcal{X}} f_N(x)g(x)\,d\mu(x) \right| \lesssim \|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})} \|g\|_{H_{\mathcal{D}}^{p,\infty}(\mathcal{X})},$$

which together with the density of $L_b^\infty(\mathcal{X}) \cap H_{\mathcal{D}}^{p,\infty}(\mathcal{X})$ in $H_{\mathcal{D}}^{p,\infty}(\mathcal{X})$ (see Remark 2.7 (ii)) implies that $\ell \in (H_{\mathcal{D}}^{p,\infty}(\mathcal{X}))^*$ and $\|\ell\|_{(H_{\mathcal{D}}^{p,\infty}(\mathcal{X}))^*} \lesssim \|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X})}$. Thus,

$$\mathcal{E}_{\mathcal{D}}^{1/p-1,\,1}(\mathcal{X}) \subset (H_{\mathcal{D}}^{p}(\mathcal{X}))^*. \tag{2.9}$$

We now prove that $(H_{\mathcal{D}}^{p,2}(\mathcal{X}))^* \subset \mathcal{E}_{\mathcal{D}}^{1/p-1,\,2}(\mathcal{X})$. Let $\ell \in (H_{\mathcal{D}}^{p,2}(\mathcal{X}))^*$. Since $H^{p,2}(\mathcal{X}) \subset H_{\mathcal{D}}^{p,2}(\mathcal{X})$, then $\ell \in (H^{p,2}(\mathcal{X}))^* = \mathcal{E}^{1/p-1,\,2}(\mathcal{X})$ (see Remark 2.5 and Remark 2.1 (i) and (iii)). Hence there exists $\widetilde{f} \in \mathcal{E}^{1/p-1,\,2}(\mathcal{X})$ such that for all constants C and $g \in L^2(\mathcal{X})$ satisfying that $\int_{\mathcal{X}} g(x)\,d\mu(x) = 0$ and $\mathrm{supp}\,(g)$ is bounded,

$$\ell(g) = \int_{\mathcal{X}} \widetilde{f}(x)g(x)\,d\mu(x) = \int_{\mathcal{X}} (\widetilde{f}(x) + C)g(x)\,d\mu(x), \tag{2.10}$$

and $\|\widetilde{f}\|_{\mathcal{E}^{1/p-1,\,2}(\mathcal{X})} \lesssim \|\ell\|_{(H^{p,2}(\mathcal{X}))^*} \lesssim \|\ell\|_{(H_{\mathcal{D}}^{p,2}(\mathcal{X}))^*}$. We then need to choose a suitable constant C such that $f \equiv \widetilde{f} + C \in \mathcal{E}_{\mathcal{D}}^{1/p-1,\,2}(\mathcal{X})$.

Observe that for all constants $\widetilde{C}$, $\widetilde{f}+\widetilde{C}\in\mathcal{E}^{1/p-1,\,2}(\mathcal{X})$. Then by Lemma 2.1 (i), to show $f\in\mathcal{E}_{\mathcal{D}}^{1/p-1,\,2}(\mathcal{X})$ and $\|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,2}(\mathcal{X})}\lesssim\|\ell\|_{(H_{\mathcal{D}}^{p,\,2}(\mathcal{X}))^*}$, it suffices to show that for all $B\in\mathcal{D}$,

$$|f_B|[\mu(B)]^{1-1/p}\lesssim\|\ell\|_{\left(H_{\mathcal{D}}^{p,\,2}(\mathcal{X})\right)^*}. \tag{2.11}$$

To this end, for any $B\in\mathcal{D}$, let $L^2(B)\equiv\{f\in L^2(\mathcal{X}):\ \mathrm{supp}\,(f)\subset B\}$ and $L^2_0(B)\equiv\{f\in L^2(B):\ \int_{\mathcal{X}}f(x)\,d\mu(x)=0\}$. Then for any $g\in L^2(B)$, the function $g[\mu(B)]^{1/2-1/p}\|g\|^{-1}_{L^2(B)}$ is a $(p,\,2)_{\mathcal{D}}$-atom supported in B and

$$|\ell(g)|\le\|\ell\|_{\left(H_{\mathcal{D}}^{p,\,2}(\mathcal{X})\right)^*}\|g\|_{H_{\mathcal{D}}^{p,\,2}(\mathcal{X})}\le[\mu(B)]^{1/p-1/2}\|\ell\|_{\left(H_{\mathcal{D}}^{p,\,2}(\mathcal{X})\right)^*}\|g\|_{L^2(B)},$$

which implies that $\ell\in(L^2(B))^*=L^2(B)$. By this together with the Riesz representation theorem, there exists a function $f^B\in L^2(B)$ such that for all $g\in L^2(B)$, $\ell(g)=\int_B f^B(x)g(x)\,d\mu(x)$ and

$$\left\|f^B\right\|_{L^2(B)}\le[\mu(B)]^{1/p-1/2}\|\ell\|_{\left(H_{\mathcal{D}}^{p,\,2}(\mathcal{X})\right)^*}. \tag{2.12}$$

Moreover, from this fact and (2.10), we deduce that for all $g\in L^2_0(B)$, $\int_{\mathcal{X}}[f^B(x)-\widetilde{f}(x)]g(x)\,d\mu(x)=0$, which further implies that $f^B-\widetilde{f}=0$ in $[L^2_0(B)]^*$. Recall that $[L^2_0(B)]^*=L^2(B)/\mathbb{C}$ (the space of functions $f\in L^2(B)$ modulo constant functions) and $f=0$ in $L^2(B)/\mathbb{C}$ if and only if f is a constant (see [4, p. 633]). Using these facts, we have that $f^B-\widetilde{f}$ is a constant C_B.

Now it suffices to verify that for all balls B, $S\in\mathcal{D}$, we have $C_B=C_S$. Observe that $g\equiv\{[\mu(\frac12B)]^{-1}\chi_{\frac12B}-[\mu(\frac12S)]^{-1}\chi_{\frac12S}\}$ is a multiple of certain $(p,\,2)$-atom, and $[\mu(\frac12B)]^{-1}\chi_{\frac12B}$ and $\mu(\frac12S)]^{-1}\chi_{\frac12S}$ are multiplies of $(p,\,2)_{\mathcal{D}}$-atoms. Therefore, from the fact that $f^B-C_B=\widetilde{f}=f^S-C_S$ and (2.10), it follows that

$$\begin{aligned}\ell(g)&=\ell\left(\left[\mu\left(\frac12B\right)\right]^{-1}\chi_{\frac12B}\right)-\ell\left(\left[\mu\left(\frac12S\right)\right]^{-1}\chi_{\frac12S}\right)\\&=\frac{1}{\mu(\frac12B)}\int_B f^B(x)\chi_{\frac12B}(x)\,d\mu(x)-\frac{1}{\mu(\frac12S)}\int_S f^S(x)\chi_{\frac12S}(x)\,d\mu(x)\\&=\int_{B\cup S}\widetilde{f}(x)g(x)\,d\mu(x)+C_B-C_S=\ell(g)+C_B-C_S,\end{aligned}$$

which implies that $C_B=C_S$. Denote the constant as above by $\widetilde{C}$ and define $f\equiv\widetilde{f}+\widetilde{C}$. Then by this, (2.12) and the Hölder inequality, we have that for all $B\in\mathcal{D}$,

$$|f_B|[\mu(B)]^{1-1/p}=|(f^B)_B|[\mu(B)]^{1-1/p}\lesssim\|\ell\|_{\left(H_{\mathcal{D}}^{p,\,2}(\mathcal{X})\right)^*}.$$

This implies (2.11), from which and Lemma 2.1 (i), we further deduce that $f\in\mathcal{E}_{\mathcal{D}}^{1/p-1,\,2}(\mathcal{X})$ and $\|f\|_{\mathcal{E}_{\mathcal{D}}^{1/p-1,\,2}(\mathcal{X})}\lesssim\|\ell\|_{(H_{\mathcal{D}}^{p,\,2}(\mathcal{X}))^*}$. Thus, $(H^p_{\mathcal{D}}(\mathcal{X}))^*\subset\mathcal{E}_{\mathcal{D}}^{1/p-1,\,2}(\mathcal{X})$, which together with Lemma 2.2 and (2.9) then completes the proof of Theorem 2.1.

3 Boundedness of the radial and the Poisson maximal functions

This section is devoted to the boundedness of the radial and the Poisson maximal functions from $\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})$ to $\widetilde{\mathcal{E}}_\rho^{\alpha,\,p}(\mathcal{X})$. We start with the notion of the radial maximal function.

Definition 3.1 Let ρ be an admissible function on $\mathcal{X}$ and $\{T_t\}_{t>0}$ a family of linear integral operators on $L^2(\mathcal{X})$. Moreover, assume that there exist positive constants C, γ, δ_1, δ_2, β satisfying that for all $t \in (0,\,\infty)$ and $x,\,x',\,y \in \mathcal{X}$ with $d(x,\,x') \le t/2$,

$$|T_t(x,\,y)| \le C\frac{1}{V_t(x)+V(x,\,y)}\left(\frac{t}{t+d(x,\,y)}\right)^{\gamma}\left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1}; \tag{3.1}$$

$$|T_t(x,\,y)-T_t(x',\,y)| \le C\frac{1}{V_t(x)+V(x,\,y)}\left(\frac{t}{t+d(x,\,y)}\right)^{\gamma}\left(\frac{d(x,\,x')}{t}\right)^{\beta}; \tag{3.2}$$

$$|1-T_t(1)(x)| \le C\left(\frac{t}{t+\rho(x)}\right)^{\delta_2}. \tag{3.3}$$

Let $\{T_t\}_{t>0}$ be as in Definition 3.1. For all $f \in L^1_{\rm loc}(\mathcal{X})$, the radial maximal function T^+ is defined by

$$T^+(f) \equiv \sup_{t>0}|T_t(f)|.$$

Then we have the following result.

Theorem 3.1 *Let $\alpha \in (-\infty,\,\gamma/n) \cap (-\infty,\,\min\{\beta/(2n),\,\delta_1/n,\,\delta_2/(2n)\}]$, $p \in (1,\,\infty)$ and ρ be an admissible function. If $\{T_t\}_{t>0}$ satisfies* (3.1) *through* (3.3)*, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})$, $T^+(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha,\,p}(\mathcal{X})$ and*

$$\|T^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,\,p}(\mathcal{X})} \le C\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}.$$

Proof. We only consider the case that $\alpha \in (0,\,\gamma/n) \cap (0,\,\min\{\beta/(2n),\,\delta_1/n,\,\delta_2/(2n)\}]$, the proof for $\alpha \in (-\infty, 0]$ is similar but easier. By the homogeneity of $\|\cdot\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}$ and $\|\cdot\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,\,p}(\mathcal{X})}$, we assume that $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$ and $\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})} = 1$.

Let $\mathcal{D}_\rho$ be as in Remark 2.2 (iv) and $B \equiv B(x_0,\,r) \in \mathcal{D}_\rho$. Observe that $T^+(f) \lesssim \mathrm{HL}(f)$, where for all $x \in \mathcal{X}$ and $f \in L^1_{\rm loc}(\mathcal{X})$, $\mathrm{HL}(f)$ denotes the Hardy-Littlewood maximal function of f defined by

$$\mathrm{HL}(f)(x) \equiv \sup_{B \ni x}\frac{1}{\mu(B)}\int_B |f(y)|\,d\mu(y).$$

Recall that HL is bounded on $L^p(\mathcal{X})$ for $p \in (1,\infty]$. Therefore T^+ is bounded on $L^p(\mathcal{X})$ for all $p \in (1,\infty]$. By this fact together with (2.1), we see that

$$\int_B [T^+(f\chi_{2B})(x)]^p\,d\mu(x) \lesssim \int_{2B}|f(x)|^p\,d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}. \tag{3.4}$$

If $t \in (0, r)$, then by (3.1), (2.1), the Hölder inequality and $\gamma > \alpha n$, we have

$$\begin{aligned}(3.5)\quad \left|T_t\left(f\chi_{(2B)^\complement}\right)(x)\right| &\lesssim \int_{(2B)^\complement} \frac{1}{V_t(x)+V(x,\,y)}\left(\frac{t}{t+d(x,\,y)}\right)^{\gamma} |f(y)|\,d\mu(y)\\ &\lesssim \sum_{j=1}^{\infty} 2^{-j\gamma}\left(\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}|f(y)|^p\,d\mu(y)\right)^{1/p}\\ &\lesssim \sum_{j=1}^{\infty} 2^{-j\gamma}\left[\mu\left(2^{j+1}B\right)\right]^{\alpha} \lesssim [\mu(B)]^{\alpha}\sum_{j=1}^{\infty}2^{-j(\gamma-\alpha n)} \lesssim [\mu(B)]^{\alpha}.\end{aligned}$$

Let $t \in [r, \infty)$. By (2.2), we see that for all $a \in (0, \infty)$, there exists a constant $\widetilde{C}_a \in [1, \infty)$ such that for all $x,\, y \in \mathcal{X}$ with $d(x, y) \le a\rho(x)$,

$$(3.6)\qquad \rho(y)/\widetilde{C}_a \le \rho(x) \le \widetilde{C}_a\rho(y).$$

Recall that $B \in \mathcal{D}_\rho$, which is equivalent to that $r \ge \rho(x_0)$. These facts imply that for all $x \in B$, $\rho(x) \lesssim r$. By this together with (3.1), (2.1), the Hölder inequality and the facts that $\gamma > \alpha n$ and $\delta_1 \ge \alpha n$, we have that for all $t \in [r, \infty)$ and $x \in B$,

$$\begin{aligned}\left|T_t\left(f\chi_{(2B)^\complement}\right)(x)\right| &\lesssim \int_{(2B)^\complement} \frac{|f(y)|}{V_t(x)+V(x,\,y)}\left(\frac{t}{t+d(x,\,y)}\right)^{\gamma}\left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1} d\mu(y)\\ &\lesssim \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1}\sum_{j=1}^{\infty}2^{-j\gamma}\frac{1}{V_{2^{j-1}t}(x)}\int_{d(x,\,y)<2^j t}|f(y)|\,d\mu(y)\\ &\lesssim \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1}\sum_{j=1}^{\infty}2^{-j\gamma}\left(\frac{1}{V_{2^{j+1}t}(x_0)}\int_{d(x_0,\,y)<2^{j+1}t}|f(y)|^p\,d\mu(y)\right)^{1/p}\\ &\lesssim \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1}\sum_{j=1}^{\infty}2^{-j\gamma}[V_{2^{j+1}t}(x_0)]^{\alpha}\\ &\lesssim \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1}\left(\frac{t}{r}\right)^{\alpha n}[\mu(B)]^{\alpha}\sum_{j=1}^{\infty}2^{-j(\gamma-\alpha n)} \lesssim [\mu(B)]^{\alpha}.\end{aligned}$$

Combining this and (3.5) yields that for all $t \in (0, \infty)$,

$$\int_B \left[T^+\left(f\chi_{(2B)^\complement}\right)(x)\right]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p},$$

which together with (3.4) gives us that

$$\int_B [T^+(f)(x)]^p\,d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

This also implies that $T^+(f)(x) < \infty$ for μ-a. e. $x \in \mathcal{X}$.

It remains to show that for all $B \equiv B(x_0,\, r) \notin \mathcal{D}_\rho$,

$$\int_B \left[T^+(f)(x) - \operatorname*{essinf}_B T^+(f)\right]^p d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

Let $f_1 \equiv (f - f_B)\chi_{2B}$, $f_2 \equiv (f - f_B)\chi_{(2B)^\complement}$, $B_1 \equiv \{x \in B:\ T_r^+(f)(x) \ge T_\infty^+(f)(x)\}$ and $B_2 \equiv B \setminus B_1$, where $T_r^+(f) \equiv \sup_{0<t<4r} |T_t(f)|$ and $T_\infty^+(f) \equiv \sup_{t\ge 4r} |T_t(f)|$. We have

$$\begin{aligned}
&\int_B \left[T^+(f)(x) - \operatorname*{essinf}_B T^+(f)\right]^p d\mu(x)\\
&\quad\lesssim \int_{B_1} \left[T_r^+(f)(x) - \operatorname*{essinf}_B |T_r(f)|\right]^p d\mu(x)\\
&\qquad+ \int_{B_2} \left[T_\infty^+(f)(x) - \operatorname*{essinf}_B T_\infty^+(f)\right]^p d\mu(x)\\
&\quad\lesssim \int_B [T_r^+(f_1)(x)]^p\, d\mu(x) + \mu(B) \sup_{x,\,y\in B}\sup_{0<t<4r} |T_t(f_B)(x) - T_r(f)(y)|^p\\
&\qquad+ \int_B \left[T_r^+(f_2)(x)\right]^p d\mu(x) + \mu(B) \sup_{x,\,y\in B}\sup_{t\ge 4r} |T_t(f)(x) - T_t(f)(y)|^p\\
&\quad\equiv \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 + \mathrm{E}_4.
\end{aligned}$$

By the Hölder inequality, $L^p(\mathcal{X})$-boundedness of T^+ and (2.1), we have

$$\mathrm{E}_1 \lesssim \int_{2B} |f(x) - f_B|^p\, d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}.$$

On the other hand, using (3.1), (2.1), the Hölder inequality, Lemma 2.4 (ii) and $\gamma > \alpha n$, we have that for all $t \in (0, 4r)$ and $x \in B$,

$$\begin{aligned}
|T_t(f_2)(x)| &\lesssim \int_{(2B)^\complement} \frac{1}{V_t(x) + V(x,\, z)} \left(\frac{t}{t + d(x,\, z)}\right)^\gamma |f(z) - f_B|\, d\mu(z)\\
&\lesssim \sum_{j=1}^\infty 2^{-j\gamma} \frac{1}{V_{2^{j-1}r}(x)} \int_{2^{j+1}B} [|f(z) - f_{2^{j+1}B}| + |f_B - f_{2^{j+1}B}|]\, d\mu(z)\\
&\lesssim [\mu(B)]^\alpha \sum_{j=1}^\infty 2^{-j(\gamma-\alpha n)} \lesssim [\mu(B)]^\alpha.
\end{aligned}$$

This implies that $\mathrm{E}_3 \lesssim [\mu(B)]^{1+\alpha p}$.

Similarly, by applying (3.1), (2.1) and $\gamma > \alpha n$, we have that for all $x \in B$,

$$\begin{aligned}
|T_r(f - f_B)(x)| &\lesssim \int_{\mathcal{X}} \frac{1}{V_r(x) + V(x,\, z)} \left(\frac{r}{r + d(x,\, z)}\right)^\gamma |f(z) - f_B|\, d\mu(z)\\
&\lesssim \sum_{j=0}^\infty 2^{-j\gamma} \frac{1}{V_{2^{j-1}r}(x)} \int_{2^{j+1}B} |f(z) - f_B|\, d\mu(z) \lesssim [\mu(B)]^\alpha.
\end{aligned} \tag{3.7}$$

From Lemma 2.4 (i), (3.3), $\delta_2 \ge \alpha n$ and $t < 4r \lesssim \rho(x_0)$, it follows that for all $x \in B$,

$$|f_B - T_t(f_B)(x)| = |f_B||1 - T_t(1)(x)| \lesssim [\mu(B)]^\alpha \left(\frac{t}{\rho(x_0)}\right)^{\delta_2-\alpha n} \lesssim [\mu(B)]^\alpha.$$

This together with (3.7) implies that

$$\begin{aligned}\mathrm{E}_2 &\lesssim \mu(B) \sup_{x,\, y\in B}\sup_{0<t<4r} \{|T_t(f_B)(x) - f_B|^p + |f_B - T_r(f_B)(y)|^p + |T_r(f_B - f)(y)|^p\}\\ &\lesssim [\mu(B)]^{1+\alpha p}.\end{aligned}$$

To estimate E_4, we first observe that for all $x,\, y \in B$, $\rho(x) \sim \rho(x_0) \sim \rho(y)$ (see (3.6)). By this and (3.2), we have that for all $x,\, y \in B$ and $t \in [4r, \infty)$,

$$|T_t(1)(x) - T_t(1)(y)| \lesssim \left(\frac{r}{t}\right)^{\beta}.$$

On the other hand, it follows from Lemma 2.4 (i) and (2.1) that

$$|f_{B(x_0,\, t)}| \lesssim \left(\frac{\rho(x_0)}{r}\right)^{\alpha n} [\mu(B)]^{\alpha}.$$

Then by these facts and $\alpha n \le \min\{\frac{\beta}{2}, \frac{\delta_2}{2}\}$, we obtain that for all $t \in [4r, \infty)$,

$$\begin{aligned}&|T_t(1)(x) - T_t(1)(y)||f_{B(x_0,\, t)}|\\ &\quad\lesssim \left(\frac{\rho(x_0)}{r}\right)^{\alpha n} [\mu(B)]^{\alpha}|T_t(1)(x) - T_t(1)(y)|^{\frac12}\,[|T_t(1)(x) - 1| + |1 - T_t(1)(y)|]^{\frac12}\\ &\quad\lesssim \left(\frac{\rho(x_0)}{r}\right)^{\alpha n} [\mu(B)]^{\alpha} \left(\frac{r}{\rho(x_0)}\right)^{\min\left\{\frac{\beta}{2}, \frac{\delta_2}{2}\right\}} \lesssim [\mu(B)]^{\alpha}.\end{aligned}$$

On the other hand, by (3.2), (2.1), the Hölder inequality, Lemma 2.4 (ii), $\gamma > \alpha n$ and $\beta \ge \alpha n$, we see that for all $x,\, y \in B$ and $t \in [4r, \infty)$,

$$\begin{aligned}&\left|T_t\left(f - f_{B(x_0,\, t)}\right)(x) - T_t\left(f - f_{B(x_0,\, t)}\right)(y)\right|\\ &\quad\lesssim \int_{\mathcal{X}} \left(\frac{d(x,\, y)}{t}\right)^{\beta} \frac{1}{V_t(x) + V(x,\, z)} \left(\frac{t}{t + d(x,\, z)}\right)^{\gamma} |f(z) - f_{B(x_0,\, t)}|\, d\mu(z)\\ &\quad\lesssim \left(\frac{r}{t}\right)^{\beta} \sum_{j=0}^{\infty} \frac{2^{-j\gamma}}{V_{2^{j-1}t}(x)} \int_{d(x,\, z)<2^j t} \left[|f(z) - f_{B(x_0,\, 2^{j+1}t)}| + \left|f_{B(x_0,\, t)} - f_{B(x_0,\, 2^{j+1}t)}\right|\right]\, d\mu(z)\\ &\quad\lesssim \left(\frac{r}{t}\right)^{\beta} \sum_{j=0}^{\infty} 2^{-j(\gamma-\alpha n)} [\mu(B(x_0,\, t))]^{\alpha} \lesssim [\mu(B)]^{\alpha}.\end{aligned}$$

These inequalities above lead to that

$$\begin{aligned}\mathrm{E}_4 &\lesssim \mu(B) \sup_{x,\, y\in B}\sup_{t\ge 4r} \left|T_t(f - f_{B(x_0,\, t)})(x) - T_t(f - f_{B(x_0,\, t)})(y)\right|^p\\ &\quad+ \mu(B) \sup_{x,\, y\in B}\sup_{t\ge 4r} \left[|T_t(1)(x) - T_t(1)(y)|\left|f_{B(x_0,\, t)}\right|\right]^p \lesssim [\mu(B)]^{1+\alpha p},\end{aligned}$$

which completes the proof of Theorem 3.1.

Now we consider the boundedness of the Poisson semigroup maximal operator. Let $\{T_t\}_{t>0}$ be a family of linear integral operators on $L^2(\mathcal{X})$. We always set

$$P_t \equiv \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} T_{t/(2\sqrt{s})}\, ds.$$

For all $f \in L^1_{\rm loc}(\mathcal{X})$, define the Poisson semigroup maximal operator P^+ by

$$P^+(f) \equiv \sup_{t>0} |P_t(f)|.$$

Lemma 3.1 *Assume that* $\{T_t\}_{t>0}$ *satisfies* (3.1) *through* (3.3) *with the same constants* δ_1, δ_2, β, γ *as there. Then* $\{P_t\}_{t>0}$ *also satisfies* (3.1) *through* (3.3) *with the constants* δ_1, δ_2', β' *and* γ', *where* $\delta_2' \in (0, 1) \cap (0, \delta_2]$, $\beta' \in (0, 1) \cap (0, \beta]$ *and* $\gamma' \in (0, 1) \cap (0, \gamma]$.

Proof. For all a, s, $t \in (0, \infty)$, from the fact that $t + a \le (1+s)(t/s + a)$, it follows that

$$\frac{t/s}{t/s + a} \le (1 + s^{-1}) \frac{t}{t+a}. \tag{3.8}$$

On the other hand, from (2.1), we deduce that for all x, $y \in \mathcal{X}$ and s, $t \in (0, \infty)$,

$$\begin{aligned} V_{t/s}(x) + V(x,\, y) &\sim \mu(B(x,\, t/s + d(x,\, y))) \\ &\gtrsim (1+s)^{-n} \mu(B(x,\, t + d(x,\, y))) \sim (1+s)^{-n} [V_t(x) + V(x,\, y)]. \end{aligned} \tag{3.9}$$

By (3.1), (3.8) and (3.9), we see that for all x, $y \in \mathcal{X}$,

$$\begin{aligned} |P_t(x,\, y)| &\lesssim \int_0^\infty e^{-s^2/4} T_{t/s}(x,\, y)\, ds \\ &\lesssim \int_0^\infty e^{-s^2/4} \frac{1}{V_{t/s}(x) + V(x,\, y)} \left(\frac{t/s}{t/s + d(x,\, y)} \right)^\gamma \left(\frac{\rho(x)}{t/s + \rho(x)} \right)^{\delta_1} ds \\ &\lesssim \frac{1}{V_t(x) + V(x,\, y)} \left(\frac{t}{t + d(x,\, y)} \right)^{\gamma'} \left(\frac{\rho(x)}{t + \rho(x)} \right)^{\delta_1} \\ &\quad \times \int_0^\infty e^{-s^2/4} (1+s)^{n+\delta_1} (1 + s^{-\gamma'})\, ds \\ &\lesssim \frac{1}{V_t(x) + V(x,\, y)} \left(\frac{t}{t + d(x,\, y)} \right)^{\gamma'} \left(\frac{\rho(x)}{t + \rho(x)} \right)^{\delta_1}. \end{aligned}$$

Now we prove that for all $t \in (0, \infty)$ and x, x', $y \in \mathcal{X}$ with $d(x,\, x') \le t/2$,

$$\left| P_t(x,\, y) - P_t(x',\, y) \right| \lesssim \left(\frac{d(x,\, x')}{t} \right)^{\beta'} \frac{1}{V_t(x) + V(x,\, y)} \left(\frac{t}{t + d(x,\, y)} \right)^{\gamma'}. \tag{3.10}$$

Observe that in this case, $t + d(x,\, y) \sim t + d(x',\, y)$ and $d(x,\, x') \le t/(2s)$ if and only if $s \le t/[2d(x,\, x')]$. Then (3.1) and (3.2) together with (3.8) and (3.9) yield that

$$\left| P_t(x,\, y) - P_t(x',\, y) \right|$$

$$
\begin{aligned}
&\lesssim \int_0^\infty e^{-s^2/4}\left|T_{t/s}(x,\,y)-T_{t/s}(x',\,y)\right| ds\\
&\lesssim \left[\int_0^{t/[2d(x,\,x')]}\left(\frac{d(x,\,x')}{t/s}\right)^\beta+\int_{t/[2d(x,\,x')]}^\infty\right]\frac{e^{-s^2/4}}{V_{t/s}(x)+V(x,\,y)}\left(\frac{t/s}{t/s+d(x,\,y)}\right)^\gamma ds\\
&\lesssim \left[\int_0^{t/[2d(x,\,x')]}(1+s)^{\beta'}+\int_{t/[2d(x,\,x')]}^\infty s^{\beta'}\right]e^{-s^2/4}(1+s)^n(1+s^{-\gamma'})\,ds\\
&\quad\times\left(\frac{d(x,\,x')}{t}\right)^{\beta'}\frac{1}{V_t(x)+V(x,\,y)}\left(\frac{t}{t+d(x,\,y)}\right)^{\gamma'}\\
&\lesssim \left(\frac{d(x,\,x')}{t}\right)^{\beta'}\frac{1}{V_t(x)+V(x,\,y)}\left(\frac{t}{t+d(x,\,y)}\right)^{\gamma'},
\end{aligned}
$$

which implies (3.10).

On the other hand, by (3.3) and (3.8), we see that for all $x \in \mathcal{X}$ and $t \in (0,\infty)$,

$$
\begin{aligned}
|1-P_t(1)(x)| &\lesssim \int_0^\infty e^{-s^2/4}\left|1-T_{t/s}(1)(x)\right| ds\\
&\lesssim \int_0^\infty e^{-s^2/4}\left(\frac{t/s}{t/s+\rho(x)}\right)^{\delta_2} ds\\
&\lesssim \left(\frac{t}{t+\rho(x)}\right)^{\delta_2'}\int_0^\infty e^{-s^2/4}(1+s^{-\delta_2'})\,ds \lesssim \left(\frac{t}{t+\rho(x)}\right)^{\delta_2'}.
\end{aligned}
$$

This finishes the proof of Lemma 3.1.

Theorem 3.2 *Let ρ be an admissible function and $\{T_t\}_{t>0}$ satisfy* (3.1) *through* (3.3) *with the same constants β, γ, δ_1, δ_2 as there and δ_2', β' and γ' be positive constants such that $\delta_2' \in (0,\,1)\cap(0,\,\delta_2]$, $\beta' \in (0,\,1)\cap(0,\,\beta]$ and $\gamma' \in (0,\,1)\cap(0,\,\gamma]$. Let $\alpha \in (-\infty,\,\gamma'/n)\cap(-\infty,\,\min\{\beta'/(2n),\,\delta_1/n,\,\delta_2'/(2n)\}]$ and $p \in (1,\,\infty)$. Then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})$, $P^+(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha,\,p}(\mathcal{X})$ and*

$$
\|P^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,\,p}(\mathcal{X})} \le C\|f\|_{\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})}.
$$

Proof. Notice that our assumption on $\{T_t\}_{t>0}$ and Lemma 3.1 imply that $\{P_t\}_{t>0}$ satisfies (3.1) through (3.3) with constants δ_1, δ_2', γ' and β'. By this and an argument similar to the proof of Theorem 3.1, we can prove Theorem 3.2. We omit the details by the similarity. This finishes the proof of Theorem 3.2.

Remark 3.1 (i) If $\alpha > 0$, then by Lemma 2.2 (ii), the spaces $\widetilde{\mathcal{E}}_\rho^{\alpha,\,p}(\mathcal{X})$ in Theorems 3.1 and 3.2 are exactly the spaces $\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})$. If $\alpha < 0$ and $\mathcal{X}$ is an RD-space, then by Remark 2.3 (ii) and the fact that the maximal operators are nonnegative, we know that if the space $\widetilde{\mathcal{E}}_\rho^{\alpha,\,p}(\mathcal{X})$ in Theorems 3.1 and 3.2 is replaced by the space $\mathcal{E}_\rho^{\alpha,\,p}(\mathcal{X})$, we obtain the same results.

(ii) Let $\mathcal{X}$ be an RD-space and ρ an admissible function. Assume that there exist constants $C \in (0,\,\infty)$, $\epsilon_1 \in (0,1]$, $\epsilon_2 \in (0,\,\infty)$, $\delta \in (0,\,1]$ and $\gamma \in (0,\,\infty)$, and an

(ϵ_1, ϵ_2)-AOTI $\{\widetilde{T}_t\}_{t>0}$ (see, for example, [12, 32] for the definition of AOTI) with kernels $\{\widetilde{T}_t(x,y)\}_{t>0}$ such that for all $t \in (0,\infty)$ and $x,\, y \in \mathcal{X}$,

$$\left|T_t(x,\, y) - \widetilde{T}_t(x,\, y)\right| \le C\left(\frac{t}{t+\rho(x)}\right)^{\delta} \frac{1}{V_t(x)+V(x,\, y)} \left(\frac{t}{t+d(x,\, y)}\right)^{\gamma}. \tag{3.11}$$

If $\alpha = 0$ and (3.1) through (3.3) were replaced by (3.11), Theorems 3.1 and 3.2 were obtained in [32]. We remark that since for all $x \in \mathcal{X}$, $\widetilde{T}_t(1)(x) = 1$ (see [32]), (3.11) implies (3.3) with $\delta_2 = \delta$.

4 Boundedness of the Littlewood-Paley g-function

In this section, we consider the boundedness of certain variant of the Littlewood-Paley g-function from $\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})$ to $\widetilde{\mathcal{E}}^{\alpha,\,p}_{\rho}(\mathcal{X})$. The boundedness from $\mathrm{BMO}_\rho(\mathcal{X})$ to $\mathrm{BLO}_\rho(\mathcal{X})$ where $\mathcal{X}$ is an RD-space of this operator was obtained in [32].

Let ρ be an admissible function on $\mathcal{X}$ and $\{Q_t\}_{t>0}$ a family of operators bounded on $L^2(\mathcal{X})$ with integral kernels $\{Q_t(x,\, y)\}_{t>0}$ satisfying that there exist constants $C \in (0,\, \infty)$, $\delta_1 \in (0,\, \infty)$, $\delta_2 \in (0,\, 1)$, $\beta \in (0,\, 1]$ and $\gamma \in (0,\, \infty)$ such that for all $t \in (0,\, \infty)$ and $x,\, x',\, y \in \mathcal{X}$ with $d(x,\, x') \le \frac{t}{2}$,

$(Q)_{\rm i}$ $|Q_t(x,\, y)| \le C\frac{1}{V_t(x)+V(x,y)}(\frac{t}{t+d(x,y)})^{\gamma}(\frac{\rho(x)}{t+\rho(x)})^{\delta_1}$;

$(Q)_{\rm ii}$ $|Q_t(x,\, y) - Q_t(x',\, y)| \le C(\frac{d(x,x')}{t+d(x,y)})^{\beta}\frac{1}{V_t(x)+V(x,y)}(\frac{t}{t+d(x,y)})^{\gamma}$;

$(Q)_{\rm iii}$ $|\int_{\mathcal{X}} Q_t(x,\, y)d\mu(y)| \le C(\frac{t}{t+\rho(x)})^{\delta_2}$.

For all $f \in L^1_{\rm loc}(\mathcal{X})$ and $x \in \mathcal{X}$, define the Littlewood-Paley g-function by

$$g(f)(x) \equiv \left(\int_0^\infty |Q_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2}. \tag{4.1}$$

Lemma 4.1 *Let $\alpha \in (-\infty, \min\{\gamma/n,\, \delta_2/n\})$, $p \in (1,\infty)$ and ρ be an admissible function on $\mathcal{X}$. Then there exists a positive constant C such that for all $f \in \mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})$,*

(i) for all $x \in \mathcal{X}$ and $t > 0$,

$$|Q_t(f)(x)| \le C\left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1} [\mu(B(x,\, t))]^{\alpha} \|f\|_{\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})};$$

(ii) for all $x,\, y \in \mathcal{X}$ and $t \ge 2d(x,\, y)$,

$$|Q_t(f)(x) - Q_t(f)(y)| \le \begin{cases} C\left(\frac{d(x,y)}{t}\right)^{\beta}\left(1+\frac{\rho(x)}{t}\right)^{\alpha n} [\mu(B(x,\, t))]^{\alpha}\|f\|_{\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})}, & \alpha > 0; \\ C\left(\frac{d(x,y)}{t}\right)^{\beta}\left(1+\log\frac{\rho(x)}{t}\right) [\mu(B(x,\, t))]^{\alpha}\|f\|_{\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})}, & \alpha \le 0. \end{cases}$$

Proof. By the homogeneity of $\|\cdot\|_{\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})}$, we may assume that $f \in \mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})$ and $\|f\|_{\mathcal{E}^{\alpha,\,p}_{\rho}(\mathcal{X})} = 1$. By $(\mathrm{Q})_{\rm i}$, (4.2), (2.1), $\gamma > \alpha n$ and the Hölder inequality, we have that for all $x \in \mathcal{X}$ and $t \ge \rho(x)$,

$$|Q_t(f)(x)| \lesssim \int_{\mathcal{X}} \frac{1}{V_t(x)+V(x,\, y)}\left(\frac{t}{t+d(x,\, y)}\right)^{\gamma}\left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1} |f(y)|\, d\mu(y) \tag{4.2}$$

$$
\begin{aligned}
&\lesssim \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1} \sum_{j=0}^{\infty} 2^{-j\gamma} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x,\,y)<2^j t} |f(y)|\,d\mu(y)\\
&\lesssim \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1} \sum_{j=0}^{\infty} 2^{-j\gamma} [\mu(B(x,\,2^j t))]^{\alpha}\\
&\lesssim \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1} [\mu(B(x,\,t))]^{\alpha} \sum_{j=0}^{\infty} \max\left\{2^{-j(\gamma-\alpha n)},\ 2^{-j\gamma}\right\}\\
&\lesssim \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1} [\mu(B(x,\,t))]^{\alpha}.
\end{aligned}
$$

Let $x \in \mathcal{X}$ and $t < \rho(x)$. In this case, $t + \rho(x) \sim \rho(x)$. Using $\gamma > \alpha n$, $(\mathrm{Q})_{\mathrm{i}}$, (2.1), Lemma 2.4 (ii) and the Hölder inequality, we have

$$
\begin{aligned}
&\left|Q_t\left(f - f_{B(x,\,t)}\right)(x)\right|\\
&\quad\lesssim \sum_{j=0}^{\infty} 2^{-j\gamma} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x,\,y)<2^j t} |f(y) - f_{B(x,\,t)}|\,d\mu(y)\\
&\quad\lesssim \sum_{j=0}^{\infty} 2^{-j\gamma} \left\{\frac{1}{V_{2^j t}(x)} \int_{d(x,\,y)<2^j t} \left|f(y) - f_{B(x,\,2^j t)}\right|\,d\mu(y) + \left|f_{B(x,\,2^j t)} - f_{B(x,\,t)}\right|\right\}\\
&\quad\lesssim [\mu(B(x,\,t))]^{\alpha} \sum_{j=0}^{\infty} 2^{-j\gamma} \max\left\{2^{j\max\{\alpha n,\,0\}}, j+1\right\} \lesssim [\mu(B(x,\,t))]^{\alpha}.
\end{aligned}
$$

On the other hand, from $(\mathrm{Q})_{\mathrm{iii}}$, Lemma 2.4 (i), $t < \rho(x)$, and the fact $\delta_2 > \alpha n$, we deduce that

$$
\begin{aligned}
|Q_t(f_{B(x,\,t)})(x)| &\lesssim [\mu(B(x,\,t))]^{\alpha} \left(\frac{t}{t+\rho(x)}\right)^{\delta_2} \max\left\{1 + \log\frac{\rho(x)}{t}, \left(\frac{\rho(x)}{t}\right)^{\max\{\alpha n,\,0\}}\right\}\\
&\lesssim [\mu(B(x,\,t))]^{\alpha} \left(\frac{\rho(x)}{t+\rho(x)}\right)^{\delta_1}.
\end{aligned}
$$

This gives (i).

To show (ii), by $(\mathrm{Q})_{\mathrm{ii}}$, we see that for all $x,\,y \in \mathcal{X}$ and $t \ge 2d(x,\,y)$,

$$
\begin{aligned}
&|Q_t(f)(x) - Q_t(f)(y)|\\
&\quad\lesssim \int_{\mathcal{X}} \left(\frac{d(x,\,y)}{t+d(x,\,z)}\right)^{\beta} \frac{1}{V_t(x)+V(x,\,z)} \left(\frac{t}{t+d(x,\,z)}\right)^{\gamma} |f(z)|\,d\mu(z)\\
&\quad\lesssim \left(\frac{d(x,\,y)}{t}\right)^{\beta} \sum_{j=0}^{\infty} 2^{-j\gamma} \frac{1}{V_{2^{j-1}t}(x)} \int_{d(x,\,z)<2^j t} |f(z)|\,d\mu(z).
\end{aligned}
\tag{4.3}
$$

Now we consider the following two cases. *Case (i)* $\alpha \in (0, \infty)$. In this case, if $t \ge \rho(x)$, by $\gamma > \alpha n$, the Hölder inequality, (4.3) and (2.1), we have

$$
|Q_t(f)(x) - Q_t(f)(y)| \lesssim \left(\frac{d(x,\,y)}{t}\right)^{\beta} \sum_{j=0}^{\infty} 2^{-j\gamma} [\mu(B(x,\,2^j t))]^{\alpha}
\tag{4.4}
$$

$$\lesssim \left(\frac{d(x,\,y)}{t}\right)^{\beta}[\mu(B(x,\,t))]^{\alpha}.$$

Assume that $t < \rho(x)$. Let $N_1 \in \mathbb{N}$ such that $2^{N_1-1}t < \rho(x) \le 2^{N_1}t$. From the Hölder inequality and (2.1), it follows that

$$\begin{aligned}(4.5)\qquad &\sum_{j=N_1}^{\infty} 2^{-j\gamma}\frac{1}{V_{2^{j-1}t}(x)}\int_{d(x,\,z)<2^jt}|f(z)|\,d\mu(z)\\ &\quad\lesssim \sum_{j=N_1}^{\infty} 2^{-j\gamma}[\mu(B(x,\,2^jt))]^{\alpha} \lesssim [\mu(B(x,\,t))]^{\alpha}.\end{aligned}$$

By the Hölder inequality, (2.1) and Lemma 2.4 (i), we see that for all $j \in \{0,1,\cdots,N_1-1\}$,

$$\frac{1}{V_{2^{j-1}t}(x)}\int_{d(x,\,z)<2^jt}|f(z)|\,d\mu(z) \lesssim \left(\frac{\rho(x)}{t}\right)^{\alpha n}[\mu(B(x,\,t))]^{\alpha}.$$

This together with $\gamma > \alpha n$ gives us that

$$\sum_{j=0}^{N_1-1} 2^{-j\gamma}\frac{1}{V_{2^{j-1}t}(x)}\int_{d(x,\,z)<2^jt}|f(z)|\,d\mu(z) \lesssim \left(\frac{\rho(x)}{t}\right)^{\alpha n}[\mu(B(x,\,t))]^{\alpha}.$$

Combining this, (4.3) through (4.5) leads to that for all $x,\,y \in \mathcal{X}$ and $t \ge 2d(x,y)$,

$$|Q_t(f)(x) - Q_t(f)(y)| \lesssim \left(1 + \frac{\rho(x)}{t}\right)^{\alpha n}[\mu(B(x,\,t))]^{\alpha}.$$

Case (ii) $\alpha \in (-\infty, 0]$. If $t \ge \rho(x)$, then (4.3) yields that

$$|Q_t(f)(x) - Q_t(f)(y)| \lesssim \left(\frac{d(x,\,y)}{t}\right)^{\beta}[\mu(B(x,\,t))]^{\alpha}.$$

Let $t < \rho(x)$ and N_1 be the integer as in Case (i). Then by (4.3), (2.1), Lemma 2.4 (i) and the Hölder inequality, we have

$$\begin{aligned}&|Q_t(f)(x) - Q_t(f)(y)|\\ &\quad\lesssim \left(\frac{d(x,\,y)}{t}\right)^{\beta}\left\{\sum_{j=0}^{N_1-1} 2^{-j\gamma}\frac{1}{V_{2^{j-1}t}(x)}\int_{d(x,\,z)<2^jt}|f(z)|\,d\mu(z) + \sum_{j=N_1}^{\infty}\cdots\right\}\\ &\quad\lesssim \left(\frac{d(x,\,y)}{t}\right)^{\beta}\left\{\sum_{j=0}^{N_1-1} 2^{-j\gamma}\left(1 + \log\frac{\rho(x)}{t}\right) + \sum_{j=N_1}^{\infty} 2^{-j\gamma}\right\}[\mu(B(x,\,t))]^{\alpha}\\ &\quad\lesssim \left(\frac{d(x,\,y)}{t}\right)^{\beta}\left(1 + \log\frac{\rho(x)}{t}\right)[\mu(B(x,\,t))]^{\alpha},\end{aligned}$$

which implies (ii) and then completes the proof of Lemma 4.1.

Theorem 4.1 *Let $p \in (1, \infty)$, ρ be an admissible function on $\mathcal{X}$, g as in (4.1) and*

$$\alpha \in (-\infty, \beta/(3n)] \cap (-\infty, \min\{\gamma/n, \delta_1/n, \delta_2/(3n)\}).$$

If $g(\cdot)$ is bounded on $L^p(\mathcal{X})$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$, $[g(f)]^2 \in \widetilde{\mathcal{E}}_\rho^{2\alpha, p/2}(\mathcal{X})$ and $\|[g(f)]^2\|_{\widetilde{\mathcal{E}}_\rho^{2\alpha, p/2}(\mathcal{X})} \le C\|f\|^2_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})}$.

Proof. By similarity, we only prove the case when $\alpha > 0$. Let $f \in \mathcal{E}_\rho^{\alpha, p}(\mathcal{X})$. By the homogeneity of $\|\cdot\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})}$ and $\|\cdot\|_{\widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathcal{X})}$, we may assume that $\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathcal{X})} = 1$. For all balls $B \equiv B(x_0, r) \in \mathcal{D}_\rho$, we need to prove that

$$\int_B [g(f)(x)]^p \, d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}. \tag{4.6}$$

For all $x \in \mathcal{X}$, write

$$[g(f)(x)]^2 = \int_0^{8r} |Q_t(f)(x)|^2 \frac{dt}{t} + \int_{8r}^\infty |Q_t(f)(x)|^2 \frac{dt}{t} \equiv [g_1(f)(x)]^2 + [g_2(f)(x)]^2.$$

By the $L^p(\mathcal{X})$-boundedness of g and (2.1), we have

$$\int_B [g_1(f\chi_{2B})(x)]^p \, d\mu(x) \lesssim \int_{2B} |f(x)|^p \, d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}. \tag{4.7}$$

By $(\mathrm{Q})_{\mathrm{i}}$, $\gamma > \alpha n$, (2.1) and the Hölder inequality, we have that for all $x \in B$ and $t < 8r$,

$$\begin{aligned}
\left|Q_t\left(f\chi_{(2B)^\complement}\right)(x)\right| &\lesssim \int_{(2B)^\complement} \frac{1}{V_t(x) + V(x, y)} \left(\frac{t}{t + d(x, y)}\right)^\gamma |f(y)| \, d\mu(y) \\
&\lesssim \left(\frac{t}{r}\right)^\gamma \sum_{j=1}^\infty 2^{-j\gamma} \frac{1}{\mu(2^{j+1}B)} \int_{2^{j+1}B} |f(y)| \, d\mu(y) \\
&\lesssim \left(\frac{t}{r}\right)^\gamma [\mu(B)]^\alpha \sum_{j=1}^\infty 2^{-j(\gamma - \alpha n)} \lesssim \left(\frac{t}{r}\right)^\gamma [\mu(B)]^\alpha.
\end{aligned}$$

From this, it follows that

$$\int_B \left[g_1\left(f\chi_{(2B)^\complement}\right)(x)\right]^p \, d\mu(x) \lesssim \left(\int_0^{8r} \left(\frac{t}{r}\right)^{2\gamma} \frac{dt}{t}\right)^{p/2} [\mu(B)]^{1+\alpha p} \lesssim [\mu(B)]^{1+\alpha p}. \tag{4.8}$$

Combining (4.7) and (4.8) leads to that

$$\int_B [g_1(f)(x)]^p \, d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}. \tag{4.9}$$

Applying Lemma 2.1 (ii) and (iii) in [33], we have that for all $x, y \in \mathcal{X}$,

$$\frac{1}{\rho(x)} \gtrsim \frac{1}{\rho(y)} \left(1 + \frac{d(x, y)}{\rho(y)}\right)^{-\frac{k_0}{(1+k_0)}},$$

where k_0 is as in Definition 2.2. By this fact, we obtain that for all $x \in B$ and $t \geq 8r$,

$$\frac{1}{\rho(x)} \gtrsim \frac{1}{\rho(x_0)}\left(1+\frac{r}{\rho(x_0)}\right)^{-\frac{k_0}{(1+k_0)}} \gtrsim \frac{1}{\rho(x_0)}\left(\frac{r}{\rho(x_0)}\right)^{-\frac{k_0}{(1+k_0)}}.$$

From this, Lemma 4.1 (i) and (2.1), we deduce that for all $x \in B$,

$$|Q_t(f)(x)| \lesssim \left(\frac{\rho(x)}{t}\right)^{\delta_1}[\mu(B(x,\,t))]^{\alpha} \lesssim \left(\frac{\rho(x_0)}{t}\right)^{\delta_1}\left(\frac{r}{\rho(x_0)}\right)^{\delta_1\frac{k_0}{(1+k_0)}}\left(\frac{t}{r}\right)^{\alpha n}[\mu(B)]^{\alpha},$$

which together with the assumption that $\delta_1 > \alpha n$ implies that

$$\begin{aligned}\int_B [g_2(f)(x)]^p\,d\mu(x) &\lesssim [\mu(B)]^{1+\alpha p}\left(\frac{r}{\rho(x_0)}\right)^{p\delta_1\frac{k_0}{(1+k_0)}}\left\{\int_{8r}^{\infty}\left(\frac{\rho(x_0)}{t}\right)^{2\delta_1}\left(\frac{t}{r}\right)^{2\alpha n}\frac{dt}{t}\right\}^{p/2}\\ &\lesssim [\mu(B)]^{1+\alpha p}\left(\frac{r}{\rho(x_0)}\right)^{p\delta_1\frac{k_0}{(1+k_0)}}\left(\frac{\rho(x_0)}{r}\right)^{p\delta_1} \lesssim [\mu(B)]^{1+\alpha p}.\end{aligned}$$

This together with (4.9) gives (4.6). Moreover, it follows from (4.6) that $g(f)(x) < \infty$ for a. e. $x \in \mathcal{X}$.

Now we assume that $B \equiv B(x_0,\,r) \notin \mathcal{D}_\rho$. We need to prove that

$$\int_B \left\{[g(f)(x)]^2 - \operatorname*{essinf}_B\,[g(f)]^2\right\}^{p/2} d\mu(x) \lesssim [\mu(B)]^{1+\alpha p}. \tag{4.10}$$

To this end, write

$$\begin{aligned}[g(f)(x)]^2 &= \int_0^{8r} |Q_t(f)(x)|^2\frac{dt}{t} + \int_{8r}^{8\rho(x_0)} \cdots + \int_{8\rho(x_0)}^{\infty} \cdots\\ &\equiv [g_r(f)(x)]^2 + [g_{r,\,\rho(x_0)}(f)(x)]^2 + [g_{\rho(x_0),\,\infty}(f)(x)]^2.\end{aligned}$$

Then

$$\begin{aligned}&\int_B \left\{[g(f)(x)]^2 - \operatorname*{essinf}_B\,[g(f)]^2\right\}^{p/2} d\mu(x)\\ &\lesssim \int_B [g_r(f)(x)]^p\,d\mu(x) + \int_B \left\{[g_{r,\,\rho(x_0)}(f)(x)]^2 - \operatorname*{essinf}_B\,[g_{r,\,\rho(x_0)}(f)]^2\right\}^{p/2} d\mu(x)\\ &\quad + \int_B \left\{[g_{\rho(x_0),\infty}(f)(x)]^2 - \operatorname*{essinf}_B\,[g_{\rho(x_0),\,\infty}(f)]^2\right\}^{p/2} d\mu(x)\\ &\lesssim \int_B [g_r(f)(x)]^p\,d\mu(x) + \mu(B)\sup_{x,\,y\in B}\left|[g_{r,\,\rho(x_0)}(f)(x)]^2 - [g_{r,\,\rho(x_0)}(f)(y)]^2\right|^{p/2}\\ &\quad + \mu(B)\sup_{x,\,y\in B}\left|[g_{\rho(x_0),\,\infty}(f)(x)]^2 - [g_{\rho(x_0),\,\infty}(f)(y)]^2\right|^{p/2} \equiv \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.\end{aligned}$$

Write $f=(f-f_B)\chi_{2B}+(f-f_B)\chi_{(2B)^\complement}+f_B\equiv f_1+f_2+f_B$. By the $L^p(\mathcal{X})$-boundedness of $g(\cdot)$ and (2.1), we have

$$\int_B [g_r(f_1)(x)]^p\,d\mu(x)\lesssim\int_{2B}|f(x)-f_B|^p\,d\mu(x)\lesssim[\mu(B)]^{1+\alpha p}. \tag{4.11}$$

Using $(\mathrm{Q})_{\mathrm{i}}$, (2.1), the Hölder inequality, Lemma 2.4 (ii) and $\gamma>\alpha n$, we obtain that for all $x\in B$,

$$\begin{aligned}|Q_t(f_2)(x)|&\lesssim\int_{(2B)^\complement}\frac{1}{V_t(x)+V(x,\,y)}\left(\frac{t}{t+d(x,\,y)}\right)^\gamma|f(y)-f_B|\,d\mu(y)\\&\lesssim\left(\frac{t}{r}\right)^\gamma\sum_{j=1}^\infty 2^{-j\gamma}\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}[|f(y)-f_{2^{j+1}B}|+|f_{2^{j+1}B}-f_B|]\,d\mu(y)\\&\lesssim\left(\frac{t}{r}\right)^\gamma[\mu(B)]^\alpha\sum_{j=1}^\infty 2^{-j(\gamma-\alpha n)}\lesssim\left(\frac{t}{r}\right)^\gamma[\mu(B)]^\alpha,\end{aligned}$$

from which it follows that

$$\int_B[g_r(f_2)(x)]^p\,d\mu(x)\lesssim[\mu(B)]^{1+\alpha p}\left(\int_0^{8r}\left(\frac{t}{r}\right)^{2\gamma}\frac{dt}{t}\right)^{p/2}\lesssim[\mu(B)]^{1+\alpha p}. \tag{4.12}$$

Recall that for all $x\in B$, $\rho(x)\sim\rho(x_0)$ (see (3.6)). By this, $(\mathrm{Q})_{\mathrm{iii}}$ and Lemma 2.4 (i), we have that for all $x\in B$,

$$|Q_t(f_B)(x)|\lesssim\left(\frac{t}{t+\rho(x)}\right)^{\delta_2}|f_B|\lesssim\left(\frac{t}{\rho(x_0)}\right)^{\delta_2}\left(\frac{\rho(x_0)}{r}\right)^{\alpha n}[\mu(B)]^\alpha.$$

This together with $\delta_2>3\alpha n$ and $r<\rho(x_0)$ implies that

$$\int_B[g_r(f_B)(x)]^p\,d\mu(x)\lesssim[\mu(B)]^{1+\alpha p}\left(\frac{\rho(x_0)}{r}\right)^{\alpha pn}\left(\int_0^{8r}\left(\frac{t}{\rho(x_0)}\right)^{2\delta_2}\frac{dt}{t}\right)^{p/2}\lesssim[\mu(B)]^{1+\alpha p}.$$

Combining this, (4.11) and (4.12) yields $\mathrm{I}_1\lesssim[\mu(B)]^{1+\alpha p}$.

Since $\gamma>\alpha n$, by Lemma 4.1, (2.1) and $\rho(x_0)\sim\rho(x)$ for all $x\in B$, we have that for all $x,\,y\in B$ and $t\in[8\rho(x_0),\infty)$,

$$|Q_t(f)(x)-Q_t(f)(y)|\lesssim\left(\frac{d(x,\,y)}{t}\right)^\beta[\mu(B(x,\,t))]^\alpha\lesssim\left(\frac{r}{t}\right)^{\beta-\alpha n}[\mu(B)]^\alpha,$$

and

$$|Q_t(f)(x)|\lesssim\left(\frac{\rho(x_0)}{t}\right)^{\delta_1}[\mu(B(x,\,t))]^\alpha\lesssim\left(\frac{\rho(x_0)}{t}\right)^{\delta_1}\left(\frac{t}{r}\right)^{\alpha n}[\mu(B)]^\alpha.$$

By these inequalities and $\beta\ge3\alpha n$, we see that for all $x,\,y\in B$,

$$\left\{[g_{\rho(x_0),\infty}(f)(x)]^2-[g_{\rho(x_0),\infty}(f)(y)]^2\right\}$$

$$
\begin{aligned}
&\le \int_{8\rho(x_0)}^{\infty} |Q_t(f)(x)+Q_t(f)(y)||Q_t(f)(x)-Q_t(f)(y)|\,\frac{dt}{t}\\
&\le \int_{8\rho(x_0)}^{\infty} \left(\frac{\rho(x_0)}{t}\right)^{\delta_1}\left(\frac{r}{t}\right)^{\beta-2\alpha n}[\mu(B)]^{2\alpha}\,\frac{dt}{t} \lesssim [\mu(B)]^{2\alpha},
\end{aligned}
$$

which implies that $\mathrm{I}_3 \lesssim [\mu(B)]^{1+\alpha p}$.

By Lemma 4.1 (i), (2.1) and the fact that for all $x\in B$, $\rho(x_0)\sim\rho(x)$, we have that for all $t\in[8r, 8\rho(x_0))$ and $x\in B$,

$$
|Q_t(f)(x)| \lesssim [\mu(B(x,\,t))]^{\alpha} \lesssim \left(\frac{t}{r}\right)^{\alpha n}[\mu(B)]^{\alpha}.
$$

Thus the fact that $\beta\ge 3\alpha n$ implies that for all $x,\,y\in B$,

$$
\begin{aligned}
&\left\{[g_{r,\,\rho(x_0)}(f)(x)]^2-[g_{r,\,\rho(x_0)}(f)(y)]^2\right\}\\
&\quad\le \int_{8r}^{8\rho(x_0)} |Q_t(f)(x)+Q_t(f)(y)||Q_t(f)(x)-Q_t(f)(y)|\,\frac{dt}{t}\\
&\quad\lesssim [\mu(B)]^{\alpha}\int_{8r}^{8\rho(x_0)} \left(\frac{t}{r}\right)^{\alpha n}|Q_t(f)(x)-Q_t(f)(y)|\,\frac{dt}{t}.
\end{aligned}
$$

Let $t\in[8r, 8\rho(x_0))$, $x,\,y\in B$. We write

$$
\begin{aligned}
&|Q_t(f)(x)-Q_t(f)(y)|\\
&\quad\le \left|\int_{\mathcal{X}}[Q_t(x,z)-Q_t(y,z)]\,[f(z)-f_B]\,d\mu(z)\right| + |f_B|\left|\int_{\mathcal{X}}[Q_t(x,z)-Q_t(y,z)]\;d\mu(z)\right|\\
&\quad\equiv \mathrm{H}_1+\mathrm{H}_2.
\end{aligned}
$$

By $(\mathrm{Q})_{\mathrm{ii}}$, $t\in[8r, 8\rho(x_0))$, (2.1) and Lemma 2.4 (ii), we see that for all $x\in B$,

$$
\begin{aligned}
\mathrm{H}_1 &\lesssim \int_{\mathcal{X}}\left(\frac{d(x,\,y)}{t+d(x,\,z)}\right)^{\beta}\frac{1}{V_t(x)+V(x,\,z)}\left(\frac{t}{t+d(x,\,z)}\right)^{\gamma}|f(z)-f_B|\,d\mu(z)\\
&\lesssim \sum_{j=0}^{\infty}\frac{r^{\beta}t^{\gamma}}{(t+2^{j-1}r)^{\beta+\gamma}}\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}\left\{|f(z)-f_{2^{j+1}B}|+|f_{2^{j+1}B}-f_B|\right\}\,d\mu(z)\\
&\lesssim \sum_{j=0}^{\infty}\frac{r^{\beta}t^{\gamma}}{(t+2^{j}r)^{\beta+\gamma}}2^{j\alpha n}[\mu(B)]^{\alpha}.
\end{aligned}
$$

From this, we deduce that

$$
\int_{8r}^{8\rho(x_0)}\left(\frac{t}{r}\right)^{\alpha n}\mathrm{H}_1\,\frac{dt}{t} \lesssim [\mu(B)]^{\alpha}\sum_{j=0}^{\infty}2^{j\alpha n}\int_{8r}^{8\rho(x_0)}\left(\frac{r}{t}\right)^{\beta-\alpha n}\frac{t^{\gamma+\beta-1}}{(t+2^{j}r)^{\beta+\gamma}}\,dt \lesssim [\mu(B)]^{\alpha}.
$$

By Lemma 2.4 (i), $(\mathrm{Q})_{\mathrm{iii}}$, $\beta\ge 3\alpha n$, $\delta_2>3\alpha n$ and the fact that for all $z\in B$, $\rho(x_0)\sim\rho(z)$, we have that for μ-a. e. $x,\,y\in B$,

$$
\int_{8r}^{8\rho(x_0)}\left(\frac{t}{r}\right)^{\alpha n}\mathrm{H}_2\,\frac{dt}{t}
$$

$$\leq \int_{8r}^{8\rho(x_0)} \left(\frac{t}{r}\right)^{\alpha n} \left(\frac{\rho(x_0)}{r}\right)^{\alpha n} [\mu(B)]^\alpha \left|Q_t(1)(x) - Q_t(1)(y)\right|^{\frac{2}{3}} \left(\frac{t}{\rho(x_0)}\right)^{\frac{\delta_2}{3}} \frac{dt}{t}$$
$$\lesssim \int_{8r}^{8\rho(x_0)} \left(\frac{\rho(x_0)}{r}\right)^{\alpha n} [\mu(B)]^\alpha \left(\frac{r}{t}\right)^{\frac{\beta}{3}} \left(\frac{t}{\rho(x_0)}\right)^{\frac{\delta_2}{3}} \frac{dt}{t}$$
$$\lesssim \int_{8r}^{\rho(x_0)} [\mu(B)]^\alpha \left(\frac{t}{\rho(x_0)}\right)^{\frac{\delta_2}{3}-\alpha n} \frac{dt}{t} \lesssim [\mu(B)]^\alpha.$$

This finishes the proof of Theorem 4.1.

As a consequence of Theorem 4.1, we have the following conclusion.

Corollary 4.1 *With the assumptions same as in Theorem 4.1, there exists a positive constant C such for all $f \in \mathcal{E}_\rho^{\alpha,p}(\mathcal{X})$, $g(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})$ and $\|g(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,p}(\mathcal{X})} \leq C\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}$.*

Proof. Since

$$0 \leq g(f) - \operatorname*{essinf}_B g(f) \leq \left\{[g(f)]^2 - \operatorname*{essinf}_B [g(f)]^2\right\}^{1/2},$$

applying (4.10), we have that for all balls $B \notin \mathcal{D}_\rho$,

$$(4.13) \left\{\frac{1}{[\mu(B)]^{1+\alpha p}} \int_B \left[g(f)(x) - \operatorname*{essinf}_B g(f)\right]^p d\mu(x)\right\}^{1/p}$$
$$\lesssim \left\{\frac{1}{[\mu(B)]^{1+\alpha p}} \int_B \left\{[g(f)(x)]^2 - \operatorname*{essinf}_B [g(f)]^2\right\}^{p/2} d\mu(x)\right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})}.$$

On the other hand, by (4.6), we obtain that for all balls $B \in \mathcal{D}_\rho$,

$$\left\{\frac{1}{[\mu(B)]^{1+\alpha p}} \int_B [g(f)(x)]^p \, d\mu(x)\right\}^{1/p} \lesssim \|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathcal{X})},$$

which together with (4.13) completes the proof of Corollary 4.1.

Remark 4.1 (i) If $\alpha = 0$ and $\mathcal{X}$ is an RD-space, Theorem 4.1 and Corollary 4.1 were already obtained in [32].

(ii) We point out that Remark 3.1 (i) is also suitable to Theorem 4.1 and Corollary 4.1.

5 Several applications

This section is divided into Subsections 5.1 through 5.4. We apply the results obtained in Sections 3 and 4, respectively, to the Schrödinger operator or the degenerate Schrödinger operator on $\mathbb{R}^d$, the sub-Laplace Schrödinger operator on Heisenberg groups or on connected and simply connected nilpotent Lie groups.

5.1 Schrödinger operators on $\mathbb{R}^d$

Let $d \geq 3$ and $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with the Euclidean norm $|\cdot|$ and the Lebesgue measure dx. Denote the Laplacian $\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^d$ by Δ and the corresponding heat (Gauss) semigroup $\{e^{t\Delta}\}_{t>0}$ by $\{\widetilde{T}_t\}_{t>0}$. Let V be a nonnegative locally integrable function on $\mathbb{R}^d$, $\mathcal{L} \equiv -\Delta + V$ the Schrödinger operator and $\{T_t\}_{t>0}$ the corresponding semigroup with kernels $\{T_t(x, y)\}_{t>0}$. Moreover, for all $t > 0$ and $x, y \in \mathbb{R}^d$, set

$$Q_t(x, y) \equiv t^2 \frac{dT_s(x, y)}{ds}\bigg|_{s=t^2}.$$

Let $q \in (d/2, d]$, $V \in \mathcal{B}_q(\mathbb{R}^d, |\cdot|, dx)$ and ρ be as in (2.3). Then we have the following estimates; see [8, 7, 9].

Proposition 5.1 *Let $q \in (d/2, d]$, $\beta \in (0, 2 - d/q)$ and $N \in \mathbb{N}$. Then there exist positive constants $\widetilde{C}$ and C, where C is independent of N, such that for all $t \in (0, \infty)$ and $x, x', y \in \mathcal{X}$ with $d(x, x') \leq \sqrt{t}/2$,*

(i) $|T_t(x, y)| \leq \widetilde{C} t^{-d/2} \exp\{-\frac{|x-y|^2}{Ct}\}[\frac{\rho(x)}{\sqrt{t}+\rho(x)}]^N[\frac{\rho(y)}{\sqrt{t}+\rho(y)}]^N$,

(ii) $|T_t(x, y) - T_t(x', y)| \leq \widetilde{C}[\frac{|x-x'|}{\sqrt{t}}]^\beta t^{-d/2} \exp\{-\frac{|x-y|^2}{Ct}\}[\frac{\rho(x)}{\sqrt{t}+\rho(x)}]^N[\frac{\rho(y)}{\sqrt{t}+\rho(y)}]^N$,

(iii) $|T_t(x, y) - \widetilde{T}_t(x, y)| \leq \widetilde{C}[\frac{\sqrt{t}}{\sqrt{t}+\rho(x)}]^{2-d/q} t^{-d/2} \exp\{-\frac{|x-y|^2}{Ct}\}$;

and for all $t \in (0, \infty)$ and $x, x', y \in \mathcal{X}$ with $d(x, x') \leq t/2$,

(iv) $|Q_t(x, y)| \leq \widetilde{C} t^{-d} \exp\{-\frac{|x-y|^2}{Ct^2}\}[\frac{\rho(x)}{t+\rho(x)}]^N[\frac{\rho(y)}{t+\rho(y)}]^N$,

(v) $|Q_t(x, y) - Q_t(x', y)| \leq \widetilde{C}[\frac{|x-x'|}{t}]^\beta t^{-d} \exp\{-\frac{|x-y|^2}{Ct^2}\}[\frac{\rho(x)}{t+\rho(x)}]^N[\frac{\rho(y)}{t+\rho(y)}]^N$,

(vi) $|\int_{\mathbb{R}^d} Q_t(x, y) d\mu(y)| \leq \widetilde{C}[\frac{t}{\rho(x)}]^{2-d/q}[\frac{\rho(x)}{t+\rho(x)}]^N$.

Let $q_1, q_2 \in (d/2, \infty]$ with $q_1 < q_2$. Observe that $\mathcal{B}_{q_2}(\mathbb{R}^d) \subset \mathcal{B}_{q_1}(\mathbb{R}^d)$. Therefore, Proposition 5.1 holds for all $q \in (d/2, \infty]$. On the other hand, recall that $\{\widetilde{T}_{t^2}\}_{t>0}$ satisfies that for all $t \in (0, \infty)$, $\widetilde{T}_{t^2}(1) = 1$ (see [8, 7]). Thus $\{T_{t^2}\}_{t>0}$ satisfies the assumptions (3.1) through (3.3). Moreover, the $L^2(\mathbb{R}^d)$-boundedness of g-function $g(\cdot)$ was obtained in [8]. Using this, (iv) and (v) of Proposition 5.1 and the vector-valued Calderón-Zygmund theory (see, for example, [25]), we obtain the $L^p(\mathbb{R}^d)$-boundedness of $g(\cdot)$ for $p \in (1, \infty)$. Then by applying this fact and Proposition 5.1, Theorems 3.1, 3.2, 4.1 and Corollary 4.1, we have the following result.

Proposition 5.2 *Let $q \in (d/2, \infty]$, $p \in (1, \infty)$, $V \in \mathcal{B}_q(\mathbb{R}^d, |\cdot|, dx)$ and ρ be as in* (2.3).

(i) If $\alpha \in (-\infty, 1/d - 1/(2q))$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathbb{R}^d)$, $T^+(f)$, $P^+(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{R}^d)$ and

$$\|T^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{R}^d)} + \|P^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{R}^d)} \leq C\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathbb{R}^d)}.$$

(ii) If $\alpha \in (-\infty, 2/(3d) - 1/(3q))$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathbb{R}^d)$, $[g(f)]^2 \in \widetilde{\mathcal{E}}_\rho^{2\alpha, p/2}(\mathbb{R}^d)$ with $\|[g(f)]^2\|_{\widetilde{\mathcal{E}}_\rho^{2\alpha, p/2}(\mathbb{R}^d)} \leq C\|f\|^2_{\mathcal{E}_\rho^{\alpha, p}(\mathbb{R}^d)}$, and $g(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{R}^d)$ with $\|g(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{R}^d)} \leq C\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathbb{R}^d)}$.

5.2 Degenerate Schrödinger operators on $\mathbb{R}^d$

Let $d \ge 3$ and $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with the Euclidean norm $|\cdot|$ and the Lebesgue measure dx. Recall that a nonnegative locally integrable function w is said to be an $A_2(\mathbb{R}^d)$ weight in the sense of Muckenhoupt if

$$\sup_{B\subset\mathbb{R}^d}\left\{\frac{1}{|B|}\int_B w(x)\,dx\right\}^{1/2}\left\{\frac{1}{|B|}\int_B [w(x)]^{-1}\,dx\right\}^{1/2}<\infty,$$

where the supremum is taken over all the balls in $\mathbb{R}^d$. Observe that if we set $w(E) \equiv \int_E w(x)dx$ for any measurable set E, then there exist positive constants C, Q and κ such that for all $x \in \mathbb{R}^d$, $\lambda > 1$ and $r > 0$,

$$C^{-1}\lambda^{\kappa}w(B(x,\,r)) \le w(B(x,\,\lambda r)) \le C\lambda^{Q}w(B(x,\,r)),$$

namely, the measure $w(x)\,dx$ satisfies (2.1). Thus $(\mathbb{R}^d,\,|\cdot|,\,w(x)\,dx)$ is a space of homogeneous type.

Let $w \in A_2(\mathbb{R}^d)$ and $\{a_{i,\,j}\}_{1\le i,\,j\le d}$ be a real symmetric matrix function satisfying that for all $x,\,\xi \in \mathbb{R}^d$,

$$C^{-1}|\xi|^2 \le \sum_{1\le i,\,j\le d} a_{i,\,j}(x)\xi_i\overline{\xi}_j \le C|\xi|^2.$$

Then the degenerate elliptic operator $\mathcal{L}_0$ is defined by

$$\mathcal{L}_0 f(x) \equiv -\frac{1}{w(x)}\sum_{1\le i,\,j\le d}\partial_i(a_{i,\,j}(\cdot)\partial_j f)(x),$$

where $x \in \mathbb{R}^d$. Denote by $\{\widetilde{T}_t\}_{t>0} \equiv \{e^{-t\mathcal{L}_0}\}_{t>0}$ the semigroup generated by $\mathcal{L}_0$.

Let V be a nonnegative locally integrable function on $w(x)\,dx$. Define the degenerate Schrödinger operator by $\mathcal{L} \equiv \mathcal{L}_0 + V$. Then $\mathcal{L}$ generates a semigroup $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$ with kernels $\{T_t(x,\,y)\}_{t>0}$. Moreover, for all $t \in (0,\infty)$ and $x,\,y \in \mathbb{R}^d$, set

$$Q_t(x,\,y) \equiv t^2\left.\frac{dT_s(x,\,y)}{ds}\right|_{s=t^2}.$$

Let $q \in (Q/2, Q]$, $V \in \mathcal{B}_q(\mathbb{R}^d, |\cdot|, w(x)\,dx)$ and ρ be as in (2.3). Then $\{T_t(\cdot,\cdot)\}_{t>0}$ and $\{Q_t(\cdot,\cdot)\}_{t>0}$ satisfy Proposition 5.1 with $t^{-d/2}$ replaced by $[V_{\sqrt{t}}(x)]^{-1}$, t^{-d} by $[V_t(x)]^{-1}$ and d by Q. In fact, the corresponding Proposition 5.1 (i) and (iii) here were given in [8]. The proof of the corresponding Proposition 5.1 (ii) here is similar to that of Proposition 5.1; see [7]. The proofs of the corresponding Proposition 5.1 (iv), (v) and (vi) here are similar to that of Proposition 4 of [9]. We omit the details here.

Recall that $\{\widetilde{T}_{t^2}\}_{t>0}$ satisfies that for all $t \in (0,\infty)$, $\widetilde{T}_{t^2}(1) = 1$; see, for example, [13]. Thus $\{T_{t^2}\}_{t>0}$ satisfies the assumptions (3.1) through (3.3). Moreover, the $L^2(\mathbb{R}^d)$-boundedness of $g(\cdot)$ can be obtained by the same argument as in Lemma 3 of [8]. Using this, (iv) and (v) of Proposition 5.1 and the vector-valued Calderón-Zygmund theory, we obtain the $L^p(\mathbb{R}^d)$-boundedness of $g(\cdot)$ for $p \in (1,\,\infty)$. Then by applying these facts and Theorems 3.1, 3.2, 4.1 and Corollary 4.1, we have the following result.

Proposition 5.3 *Let $w \in A_2(\mathbb{R}^d)$, $q \in (Q/2, \infty]$, $p \in (1, \infty)$, $V \in \mathcal{B}_q(\mathbb{R}^d, |\cdot|, w(x)\,dx)$ and ρ be as in (2.3) with $d\mu = w(x)\,dx$.*

(i) If $\alpha \in (-\infty, 1/Q - 1/(2q))$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,p}(w(x)\,dx)$, $T^+(f)$, $P^+(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha,p}(w(x)\,dx)$ and

$$\|T^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,p}(w(x)\,dx)} + \|P^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,p}(w(x)\,dx)} \le C\|f\|_{\mathcal{E}_\rho^{\alpha,p}(w(x)\,dx)}.$$

(ii) If $\alpha \in (-\infty, 2/(3Q) - 1/(3q))$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,p}(w(x)\,dx)$, $[g(f)]^2 \in \widetilde{\mathcal{E}}_\rho^{2\alpha,p/2}(w(x)\,dx)$ with $\|[g(f)]^2\|_{\widetilde{\mathcal{E}}_\rho^{2\alpha,p/2}(w(x)\,dx)} \le C\|f\|^2_{\mathcal{E}_\rho^{\alpha,p}(w(x)\,dx)}$, and $g(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha,p}(w(x)\,dx)$ with

$$\|g(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,p}(w(x)\,dx)} \le C\|f\|_{\mathcal{E}_\rho^{\alpha,p}(w(x)\,dx)}.$$

5.3 Schrödinger operators on Heisenberg groups

The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is a connected and simply connected nilpotent Lie group with the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ and the multiplication

$$(x, s)(y, s) = \left(x + y,\, t + s + 2\sum_{j=1}^n [x_{n+j}y_j - x_j y_{n+j}]\right).$$

The homogeneous norm on $\mathbb{H}^n$ is defined by $|(x, t)| = (|x|^4 + |t|^2)^{1/4}$ for all $(x, t) \in \mathbb{H}^n$, which induces a left-invariant metric $d((x, t), (y, s)) = |(-x, -t)(y, s)|$. Moreover, there exists a positive constant C such that $|B((x, t), r)| = Cr^Q$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$ and $|B((x, t), r)|$ is the Lebesgue measure of the ball $B((x, t), r)$. The triplet $(\mathbb{H}^n, d, dx)$ is a space of homogeneous type.

A basis for the Lie algebra of left invariant vector fields on $\mathbb{H}^n$ is given by

$$X_{2n+1} = \frac{\partial}{\partial t},\quad X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t},\quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t},\ j = 1, \cdots, n.$$

All non-trivial commutators are $[X_j, X_{n+j}] = -4X_{2n+1}$, $j = 1, \cdots, n$. The sub-Laplacian has the form $\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2$.

Let V be a nonnegative locally integrable function on $\mathbb{H}^n$. Define the sub-Laplacian Schrödinger operator by $\mathcal{L} \equiv -\Delta_{\mathbb{H}^n} + V$. Denote by $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$ with kernels $\{T_t(x, y)\}_{t>0}$ and by $\{\widetilde{T}_t\}_{t>0} \equiv \{e^{t\Delta_{\mathbb{H}^n}}\}_{t>0}$. Moreover, for all $t \in (0, \infty)$ and $x, y \in \mathbb{R}^d$, set

$$Q_t(x, y) \equiv t^2 \left.\frac{dT_s(x, y)}{ds}\right|_{s=t^2}.$$

Let $V \in \mathcal{B}_q(\mathbb{H}^n, d, dx)$ with $q \in (n+1, 2n+2]$ and ρ be as in (2.3). Then $\{T_t(\cdot,\cdot)\}_{t>0}$ and $\{Q_t(\cdot,\cdot)\}_{t>0}$ satisfy Proposition 5.1 with d replaced by $2(n+2)$ and $|x - y|$ replaced by $d(x, y)$; see [21].

Observe that $\{\widetilde{T}_{t^2}\}_{t>0}$ satisfies that for all $t \in (0,\infty)$, $\widetilde{T}_{t^2}(1) = 1$ (see also [32]). Thus $\{T_{t^2}\}_{t>0}$ satisfies the assumptions (3.1) through (3.3). Moreover, the $L^2(\mathbb{H}^n)$-boundedness of $g(\cdot)$ was obtained in [21]. Using this, (iv) and (v) of Proposition 5.1 and the vector-valued Calderón-Zygmund theory, we obtain the $L^p(\mathbb{H}^n)$-boundedness of $g(\cdot)$ for $p \in (1, \infty)$. Then by applying these facts and Theorems 3.1, 3.2, 4.1 and Corollary 4.1, we have the following conclusions.

Proposition 5.4 *Let $q \in (n+1,\infty]$, $p \in (1, \infty)$, $V \in \mathcal{B}_q(\mathbb{H}^n, d, dx)$ and ρ be as in (2.3).*

(i) If $\alpha \in (-\infty, 1/(2n+2) - 1/(2q))$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,p}(\mathbb{H}^n)$, $T^+(f)$, $P^+(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha,p}(\mathbb{H}^n)$ and

$$\|T^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,p}(\mathbb{H}^n)} + \|P^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,p}(\mathbb{H}^n)} \le C\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathbb{H}^n)}.$$

(ii) If $\alpha \in (-\infty, 1/(3n+3) - 1/(3q))$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha,p}(\mathbb{H}^n)$, $[g(f)]^2 \in \widetilde{\mathcal{E}}_\rho^{2\alpha,p/2}(\mathbb{H}^n)$ with $\|[g(f)]^2\|_{\widetilde{\mathcal{E}}_\rho^{2\alpha,p/2}(\mathbb{H}^n)} \le C\|f\|^2_{\mathcal{E}_\rho^{\alpha,p}(\mathbb{H}^n)}$, and $g(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha,p}(\mathbb{H}^n)$ with $\|g(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha,p}(\mathbb{H}^n)} \le C\|f\|_{\mathcal{E}_\rho^{\alpha,p}(\mathbb{H}^n)}$.

5.4 Schrödinger operators on connected and simply connected nilpotent Lie groups

Let $\mathbb{G}$ be a connected and simply connected nilpotent Lie group and $X \equiv \{X_1, \cdots, X_k\}$ left invariant vector fields on $\mathbb{G}$ satisfying the Hörmander condition that $\{X_1, \cdots, X_k\}$ together with their commutators of order $\le m$ generates the tangent space of $\mathbb{G}$ at each point of $\mathbb{G}$. Let d be the Carnot-Carathéodory (control) distance on $\mathbb{G}$ associated to $\{X_1, \cdots, X_k\}$. Fix a left invariant Haar measure μ on $\mathbb{G}$. Then for all $x \in \mathbb{G}$, $V_r(x) = V_r(e)$; moreover, there exist κ, $D \in (0,\infty)$ with $\kappa \le D$ such that for all $x \in \mathbb{G}$, $C^{-1}r^\kappa \le V_r(x) \le Cr^\kappa$ when $r \in (0, 1]$, and $C^{-1}r^D \le V_r(x) \le Cr^D$ when $r \in (1,\infty)$; see [23] and [30]. Thus $(\mathbb{G}, d, \mu)$ is a space of homogeneous type.

The sub-Laplacian is given by $\Delta_{\mathbb{G}} \equiv \sum_{j=1}^k X_j^2$. Denote by $\{\widetilde{T}_t\}_{t>0} \equiv \{e^{t\Delta_{\mathbb{G}}}\}_{t>0}$ the semigroup generated by $-\Delta_{\mathbb{G}}$.

Let V be a nonnegative locally integrable function on $\mathbb{G}$. Then the sub-Laplace Schrödinger operator $\mathcal{L}$ is defined by $\mathcal{L} \equiv -\Delta_{\mathbb{G}} + V$. The operator $\mathcal{L}$ generates a semigroup of operators $\{T_t\}_{t>0} \equiv \{e^{-t\mathcal{L}}\}_{t>0}$, whose kernels are denoted by $\{T_t(x, y)\}_{t>0}$. Define the radial maximal operator T^+ by $T^+(f)(x) \equiv \sup_{t>0} |e^{-t\mathcal{L}}(f)(x)|$ for all $x \in \mathbb{G}$.

Let $q > D/2$, $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ and ρ be as in (2.3). For all $x, y \in \mathbb{G}$ and $t \in (0,\infty)$, define

$$Q_t(x, y) \equiv t^2 \frac{d}{ds}\Big|_{s=t^2} T_s(x, y).$$

Then $\{T_t(\cdot,\cdot)\}_{t>0}$ and $\{Q_t(\cdot,\cdot)\}_{t>0}$ satisfy Proposition 5.1 with t^{-d} replaced by $[V_t(x)]^{-1}$, $t^{-d/2}$ by $[V_{\sqrt{t}}(x)]^{-1}$ and d by D (see [33, 32]). Observe that $\{\widetilde{T}_{t^2}\}_{t>0}$ satisfies that for all $t \in (0,\infty)$, $\widetilde{T}_{t^2}(1) = 1$; see, for example, [30]. Thus $\{T_{t^2}\}_{t>0}$ satisfies the assumptions (3.1) through (3.3). Moreover, the $L^2(\mathbb{G})$-boundedness of $g(\cdot)$ can be obtained by the same argument as in Lemma 3 in [8]. Using this, (iv) and (v) of Proposition 5.4 and

the vector-valued Calderón-Zygmund theory, we obtain the $L^p(\mathbb{G})$-boundedness of $g(\cdot)$ for $p \in (1, \infty)$. Then by applying these facts and Theorems 3.1, 3.2, 4.1 and Corollary 4.1, we have the following conclusions.

Proposition 5.5 *Let $q \in (D/2, \infty]$, $p \in (1, \infty)$, $V \in \mathcal{B}_q(\mathbb{G}, d, \mu)$ and ρ be as in* (2.3).

(i) If $\alpha \in (-\infty, 1/D - 1/(2q))$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathbb{G})$, $T^+(f)$, $P^+(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{G})$ and

$$\|T^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{G})} + \|P^+(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{G})} \le C\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathbb{G})}.$$

(ii) If $\alpha \in (-\infty, 2/(3D) - 1/(3q))$, then there exists a positive constant C such that for all $f \in \mathcal{E}_\rho^{\alpha, p}(\mathbb{G})$, $[g(f)]^2 \in \widetilde{\mathcal{E}}_\rho^{2\alpha, p/2}(\mathbb{G})$ with $\|[g(f)]^2\|_{\widetilde{\mathcal{E}}_\rho^{2\alpha, p/2}(\mathbb{G})} \le C\|f\|^2_{\mathcal{E}_\rho^{\alpha, p}(\mathbb{G})}$, and $g(f) \in \widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{G})$ with $\|g(f)\|_{\widetilde{\mathcal{E}}_\rho^{\alpha, p}(\mathbb{G})} \le C\|f\|_{\mathcal{E}_\rho^{\alpha, p}(\mathbb{G})}$.

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